The extended permutohedron on a transitive relation

Luigi Santocanale and Friedrich Wehrung

LIF (Marseille) and LMNO (Caen)

E-mail (Santocanale): luigi.santocanale@lif.univ-mrs.fr
URL (Santocanale): http://www.lif.univ-mrs.fr/~lsantoca

E-mail (Wehrung): wehrung@math.unicaen.fr
URL (Wehrung): http://www.math.unicaen.fr/~wehrung

SSAOS 2012, Nový Smokovec, September 2012
What is the permutohedron?

- The **permutohedron on $n$ letters**, denoted by $P(n)$, can be defined as the set of all permutations of $n$ letters, with the ordering
What is the permutohedron?

- The **permutohedron on \( n \) letters**, denoted by \( P(n) \), can be defined as the set of all permutations of \( n \) letters, with the ordering

\[
\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta),
\]

where we set \([n] = \{1, 2, \ldots, n\}\), \(I_n = \{ (i, j) \in [n] \times [n] | i < j \}\), and \(\text{Inv}(\alpha) = \{ (i, j) \in I_n | \alpha^{-1}(i) > \alpha^{-1}(j) \}\).
What is the permutohedron?

- The permutohedron on \( n \) letters, denoted by \( P(n) \), can be defined as the set of all permutations of \( n \) letters, with the ordering

\[
\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta),
\]

where we set

\[
[n] = \{1, 2, \ldots, n\},
\]

\[
J_n = \{(i, j) \in [n] \times [n] \mid i < j\},
\]

\[
\text{Inv}(\alpha) = \{(i, j) \in J_n \mid \alpha^{-1}(i) > \alpha^{-1}(j)\}.
\]
What is the permutohedron?

- The permutohedron on \( n \) letters, denoted by \( P(n) \), can be defined as the set of all permutations of \( n \) letters, with the ordering

\[
\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta),
\]

where we set

\[
[n] = \{1, 2, \ldots, n\},
\]

\[
\mathcal{I}_n = \{(i, j) \in [n] \times [n] \mid i < j\},
\]

\[
\text{Inv}(\alpha) = \{(i, j) \in \mathcal{I}_n \mid \alpha^{-1}(i) > \alpha^{-1}(j)\}.
\]

- Alternate definition: \( P(n) = \{\text{Inv}(\sigma) \mid \sigma \in \mathfrak{S}_n\} \), ordered by \( \subseteq \).
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $J_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 

What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$.  
  \textit{(Proof:} let $(i, j) \in \mathcal{J}_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.)
What are the Inv(σ)?

- Both Inv(σ) and \( J_n \setminus \text{Inv}(\sigma) \) are transitive relations on \([n]\).
  
  \( \text{Proof:} \) let \((i, j) \in J_n \). Then \((i, j) \in \text{Inv}(\sigma) \) iff \( \sigma^{-1}(i) > \sigma^{-1}(j) \); \((i, j) \notin \text{Inv}(\sigma) \) iff \( \sigma^{-1}(i) < \sigma^{-1}(j) \).

- Conversely, every subset \( x \subseteq J_n \), such that both \( x \) and \( J_n \setminus x \) are transitive, is Inv(σ) for a unique \( \sigma \in \mathcal{S}_n \) (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{I}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 
  (*Proof:* let $(i, j) \in \mathcal{I}_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.)

- Conversely, every subset $x \subseteq \mathcal{I}_n$, such that both $x$ and $\mathcal{I}_n \setminus x$ are transitive, is $\text{Inv}(\sigma)$ for a unique $\sigma \in \mathcal{S}_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

- Say that $x \subseteq \mathcal{I}_n$ is closed if it is transitive, open if $\mathcal{I}_n \setminus x$ is closed, and clopen if it is both closed and open.
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 
  \[(\text{Proof: let } (i, j) \in \mathcal{J}_n. \text{ Then } (i, j) \in \text{Inv}(\sigma) \text{ iff } \sigma^{-1}(i) > \sigma^{-1}(j); \ (i, j) \notin \text{Inv}(\sigma) \text{ iff } \sigma^{-1}(i) < \sigma^{-1}(j).)\]

- Conversely, every subset $\mathbf{x} \subseteq \mathcal{J}_n$, such that both $\mathbf{x}$ and $\mathcal{J}_n \setminus \mathbf{x}$ are transitive, is $\text{Inv}(\sigma)$ for a unique $\sigma \in S_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

- Say that $\mathbf{x} \subseteq \mathcal{J}_n$ is closed if it is transitive, open if $\mathcal{J}_n \setminus \mathbf{x}$ is closed, and clopen if it is both closed and open.

- Hence $\mathcal{P}(n) = \{ \mathbf{x} \subseteq \mathcal{J}_n \mid \mathbf{x} \text{ is clopen} \}$, ordered by $\subseteq$. 
What are the $\text{Inv}(\sigma)$?

- Both $\text{Inv}(\sigma)$ and $\mathcal{J}_n \setminus \text{Inv}(\sigma)$ are transitive relations on $[n]$. 
  \textit{(Proof:} let $(i, j) \in \mathcal{J}_n$. Then $(i, j) \in \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) > \sigma^{-1}(j)$; $(i, j) \notin \text{Inv}(\sigma)$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$.)

- Conversely, every subset $x \subseteq \mathcal{J}_n$, such that both $x$ and $\mathcal{J}_n \setminus x$ are transitive, is $\text{Inv}(\sigma)$ for a unique $\sigma \in \mathcal{S}_n$ (Dushnik and Miller 1941, Guilbaud and Rosenstiehl 1963).

- Say that $x \subseteq \mathcal{J}_n$ is \textit{closed} if it is transitive, \textit{open} if $\mathcal{J}_n \setminus x$ is closed, and \textit{clopen} if it is both closed and open.

- Hence $\mathcal{P}(n) = \{x \subseteq \mathcal{J}_n \mid x \text{ is clopen}\}$, ordered by $\subseteq$.

- Observe that each $x \in \mathcal{P}(n)$ is a strict ordering. It can be proved (Dushnik and Miller 1941) that those are exactly the finite strict orderings of order-dimension $2$. 
The permutohedra $P(2)$, $P(3)$, and $P(4)$.

The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in Reg($e$)
Bip-Cambrian lattices
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x_c = I_n \setminus x$ defines an orthocomplementation on $P(n)$:

- $x \leq y \implies y_c \leq x_c$;
- $(x_c)_c = x$;
- $x \land x_c = 0$ (equivalently, $x \lor x_c = 1$).

Hence $P(n)$ is an ortholattice.
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$. 
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = J_n \setminus x$ defines an orthocomplementation on $P(n)$:
Permutohedra are ortholattices

**Theorem (Guilbaud and Rosenstiehl 1963)**

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = J_n \setminus x$ defines an orthocomplementation on $P(n)$:

$$x \leq y \Rightarrow y^c \leq x^c;$$

$$(x^c)^c = x;$$

$$x \land x^c = 0 \quad \text{(equivalently, } x \lor x^c = 1) .$$
Permutohedra are ortholattices

Theorem (Guilbaud and Rosenstiehl 1963)

The permutohedron $P(n)$ is a lattice, for every positive integer $n$.

The assignment $x \mapsto x^c = J_n \setminus x$ defines an orthocomplementation on $P(n)$:

- $x \leq y \Rightarrow y^c \leq x^c$;
- $(x^c)^c = x$;
- $x \land x^c = 0$ (equivalently, $x \lor x^c = 1$).

Hence $P(n)$ is an ortholattice.
What makes $P(n)$ a lattice?

- Every $x \in J_n$ is contained in a least closed set (namely, $\text{cl}(x) = \text{transitive closure of } x$).
What makes $P(n)$ a lattice?

- Every $x \in J_n$ is contained in a least closed set (namely, $\text{cl}(x) = \text{transitive closure of } x$).
- Dually, every $x \subseteq J_n$ contains a largest open set (namely, $\text{int}(x) = J_n \setminus \text{cl}(J_n \setminus x)$).
What makes $P(n)$ a lattice?

- Every $x \in J_n$ is contained in a least closed set (namely, $\text{cl}(x) = \text{transitive closure of } x$).
- Dually, every $x \subseteq J_n$ contains a largest open set (namely, $\text{int}(x) = J_n \setminus \text{cl}(J_n \setminus x)$).

**Theorem (Guilbaud and Rosenstiehl 1963)**
What makes $\mathcal{P}(n)$ a lattice?

- Every $x \in I_n$ is contained in a least closed set (namely, $\text{cl}(x) = \text{transitive closure of } x$).
- Dually, every $x \subseteq I_n$ contains a largest open set (namely, $\text{int}(x) = I_n \setminus \text{cl}(I_n \setminus x)$).

**Theorem (Guilbaud and Rosenstiehl 1963)**

$\text{int}(x)$ is closed, for any closed $x \subseteq I_n$. 
What makes $P(n)$ a lattice?

- Every $x \in J_n$ is contained in a least closed set (namely, $\text{cl}(x) = \text{transitive closure of } x$).
- Dually, every $x \subseteq J_n$ contains a largest open set (namely, $\text{int}(x) = J_n \setminus \text{cl}(J_n \setminus x)$).

**Theorem (Guilbaud and Rosenstiehl 1963)**

$\text{int}(x)$ is closed, for any closed $x \subseteq J_n$.

In particular, the join of $\{x, y\}$ in $P(n)$ is $\text{cl}(x \cup y)$. Dually, the meet of $\{x, y\}$ in $P(n)$ is $\text{int}(x \cap y)$. 
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
 Completely join-irreducible elements in Reg(e)
Bip-Cambrian lattices
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is semidistributive, for every positive integer $n$. Thus it is also pseudocomplemented.
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is semidistributive, for every positive integer $n$. Thus it is also pseudocomplemented.

**Semidistributivity** means that

\[ x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z, \text{ and, dually,} \]
\[ x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z. \]
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is semidistributive, for every positive integer $n$. Thus it is also pseudocomplemented.

Semidistributivity means that

\[ x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z, \text{ and, dually,} \]
\[ x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z. \]

Theorem (Caspard 2000)
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is \textit{semidistributive}, for every positive integer $n$. Thus it is also \textit{pseudocomplemented}.

\textbf{Semidistributivity} means that

$x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z$, and, dually,

$x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z$.

Theorem (Caspard 2000)

The permutohedron $P(n)$ is a \textit{bounded homomorphic image of a free lattice}, for every positive integer $n$. 
Permutohedra are even more peculiar lattices

Theorem (Duquenne and Cherfouh 1994, Le Conte de Poly-Barbut 1994)

The permutohedron $P(n)$ is **semidistributive**, for every positive integer $n$. Thus it is also **pseudocomplemented**.

**Semidistributivity** means that

\[ x \lor z = y \lor z \Rightarrow x \lor z = (x \land y) \lor z, \text{ and, dually,} \]
\[ x \land z = y \land z \Rightarrow x \land z = (x \lor y) \land z. \]

Theorem (Caspard 2000)

The permutohedron $P(n)$ is a **bounded homomorphic image of a free lattice**, for every positive integer $n$.

This means that there are a finitely generated free lattice $F$ and a surjective lattice homomorphism $f : F \to P(n)$ such that each $f^{-1}\{x\}$ has both a least and a largest element.
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in Reg(e)

Bip-Cambrian lattices

The definition of the permutohedron got extended to any poset $E$, in a 1995 paper by Pouzet, Reuter, Rival, and Zaguia.
The extended permutohedron

The definition of the permutohedron got extended to any poset $E$, in a 1995 paper by Pouzet, Reuter, Rival, and Zaguia.

Setting $\delta_E = \{(x, y) \in E \times E \mid x < y\}$, let $a \subseteq \delta_E$ be closed if it is transitive, open if $\delta_E \setminus a$ is closed, and clopen if it is both closed and open.
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in Reg(\(E\))

Bip-Cambrian lattices

Basic definitions

- The definition of the permutohedron got extended to any poset \(E\), in a 1995 paper by Pouzet, Reuter, Rival, and Zaguia.

- Setting \(\delta_E = \{(x, y) \in E \times E \mid x < y\}\), let \(a \subseteq \delta_E\) be closed if it is transitive, open if \(\delta_E \setminus a\) is closed, and clopen if it is both closed and open.

- Then we set

\[
\begin{align*}
P(E) &= \left\{a \subseteq \delta_E \mid a \text{ is clopen}\right\}, \quad \text{(that’s our guy)} \\
P^*(E) &= \left\{u \cap \delta_E \mid u \text{ strict linear ordering on } E\right\}.
\end{align*}
\]
The extended permutohedron got extended to any poset $E$, in a 1995 paper by Pouzet, Reuter, Rival, and Zaguia.

Setting $\delta_E = \{(x, y) \in E \times E \mid x < y\}$, let $a \subseteq \delta_E$ be closed if it is transitive, open if $\delta_E \setminus a$ is closed, and clopen if it is both closed and open.

Then we set

$P(E) = \{a \subseteq \delta_E \mid a \text{ is clopen}\}, \quad (\text{that's our guy})$

$P^*(E) = \{u \cap \delta_E \mid u \text{ strict linear ordering on } E\}.$

Obviously, $P^*(E) \subseteq P(E)$. 
The extended permutohedron

The definition of the permutohedron got extended to any poset $E$, in a 1995 paper by Pouzet, Reuter, Rival, and Zaguia.

Setting $\delta_E = \{(x, y) \in E \times E \mid x < y\}$, let $a \subseteq \delta_E$ be closed if it is transitive, open if $\delta_E \setminus a$ is closed, and clopen if it is both closed and open.

Then we set

$$P(E) = \{a \subseteq \delta_E \mid a \text{ is clopen}\} \quad \text{(that's our guy)}$$

$$P^*(E) = \{u \cap \delta_E \mid u \text{ strict linear ordering on } E\}.$$ 

Obviously, $P^*(E) \subseteq P(E)$.

Also, both $P(E)$ and $P^*(E)$ are orthocomplemented posets.
Is $P(E)$ a lattice?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

The following statements hold, for any poset $E$.

1. $P(E)$ is a lattice iff $E$ is square-free.
2. $P(E) = P^*(E)$ iff $E$ is crown-free.

Illustrating square and crowns:
Is $P(E)$ a lattice?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

The following statements hold, for any poset $E$.

1. $P(E)$ is a lattice iff $E$ is square-free.
2. $P(E) = P^*(E)$ iff $E$ is crown-free.

Illustrating square and crowns:
Is $P(E)$ a lattice?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

The following statements hold, for any poset $E$.

1. $P(E)$ is a lattice iff $E$ is square-free.
Is $P(E)$ a lattice?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

The following statements hold, for any poset $E$.

1. $P(E)$ is a lattice iff $E$ is square-free.
2. $P(E) = P^*(E)$ iff $E$ is crown-free.
Is $P(E)$ a lattice?

**Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)**

The following statements hold, for any poset $E$.

1. $P(E)$ is a lattice iff $E$ is square-free.
2. $P(E) = P^*(E)$ iff $E$ is crown-free.

Illustrating square and crowns:
What about boundedness?

Theorem (Caspard, Santocanale, and W 2011)

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$.

By invoking Caspard's 2000 theorem, we get the following extension of that result.

Corollary (Caspard, Santocanale, and W 2011)

Let $E$ be a finite square-free poset. Then $P(E)$ is a bounded homomorphic image of a free lattice.

"Square-free" is just put there in order to ensure that $P(E)$ be a lattice. For $E$ an infinite chain, $P(E)$ is not even semidistributive.
What about boundedness?

Theorem (Caspard, Santocanale, and W 2011)

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$. 

"Square-free" is just put there in order to ensure that $P(E)$ be a lattice. For $E$ an infinite chain, $P(E)$ is not even semidistributive.
What about boundedness?

**Theorem (Caspard, Santocanale, and W 2011)**

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$.

By invoking Caspard’s 2000 theorem, we get the following extension of that result.
What about boundedness?

**Theorem (Caspard, Santocanale, and W 2011)**

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$.

By invoking Caspard’s 2000 theorem, we get the following extension of that result.

**Corollary (Caspard, Santocanale, and W 2011)**
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

What about boundedness?

**Theorem (Caspard, Santocanale, and W 2011)**

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$.

By invoking Caspard’s 2000 theorem, we get the following extension of that result.

**Corollary (Caspard, Santocanale, and W 2011)**

Let $E$ be a finite square-free poset. Then $P(E)$ is a bounded homomorphic image of a free lattice.
What about boundedness?

**Theorem (Caspard, Santocanale, and W 2011)**

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$.

By invoking Caspard’s 2000 theorem, we get the following extension of that result.

**Corollary (Caspard, Santocanale, and W 2011)**

Let $E$ be a finite square-free poset. Then $P(E)$ is a bounded homomorphic image of a free lattice.

- “Square-free” is just put there in order to ensure that $P(E)$ be a lattice.
What about boundedness?

Theorem (Caspard, Santocanale, and W 2011)

Let $E$ be a square-free poset. Then the lattice $P(E)$ is a subdirect product of the $P(C)$, for all maximal chains $C$ of $E$.

By invoking Caspard’s 2000 theorem, we get the following extension of that result.

Corollary (Caspard, Santocanale, and W 2011)

Let $E$ be a finite square-free poset. Then $P(E)$ is a bounded homomorphic image of a free lattice.

- “Square-free” is just put there in order to ensure that $P(E)$ be a lattice.
- For $E$ an infinite chain, $P(E)$ is not even semidistributive.
Why is $P^*(E)$ sometimes better than $P(E)$?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

Let $E$ be a finite poset. Then the inclusion mapping from $P^*(E)$ into the powerset of $\delta E$ is cover-preserving.

Theorem (Caspard, Santocanale, and W 2011)

There is a finite poset $E$ such that the inclusion mapping from $P(E)$ into the powerset of $\delta E$ is not height-preserving (thus also not cover-preserving).

Here is the counterexample:

\[
\begin{array}{cccccccc}
x_0 & x_2 & x_4 & z_1 & z_3 & z_5 & y_{12} & y_{01} & y_{23} & y_{34} & y_{50} & y_{45}
\end{array}
\]
Why is $P^*(E)$ sometimes better than $P(E)$?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

Let $E$ be a finite poset. Then the inclusion mapping from $P^*(E)$ into the powerset of $\delta_E$ is cover-preserving.
Why is $P^*(E)$ sometimes better than $P(E)$?

**Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)**

Let $E$ be a finite poset. Then the inclusion mapping from $P^*(E)$ into the powerset of $\delta_E$ is cover-preserving.

**Theorem (Caspard, Santocanale, and W 2011)**
Why is $P^*(E)$ sometimes better than $P(E)$?

**Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)**

Let $E$ be a finite poset. Then the inclusion mapping from $P^*(E)$ into the powerset of $\delta_E$ is cover-preserving.

**Theorem (Caspard, Santocanale, and W 2011)**

There is a finite poset $E$ such that the inclusion mapping from $P(E)$ into the powerset of $\delta_E$ is not height-preserving (thus also not cover-preserving).
Why is $P^*(E)$ sometimes better than $P(E)$?

Theorem (Pouzet, Reuter, Rival, and Zaguia 1995)

Let $E$ be a finite poset. Then the inclusion mapping from $P^*(E)$ into the powerset of $\delta_E$ is cover-preserving.

Theorem (Caspard, Santocanale, and W 2011)

There is a finite poset $E$ such that the inclusion mapping from $P(E)$ into the powerset of $\delta_E$ is not height-preserving (thus also not cover-preserving).

Here is the counterexample:
Setting the problem

- Lattice-theoretical properties of $P(E)$: make sense only in case $P(E)$ is a lattice (duh), that is, $E$ is square-free.
Lattice-theoretical properties of \( P(E) \): make sense only in case \( P(E) \) is a lattice (duh), that is, \( E \) is square-free.

Is there anything left in case \( E \) is not square-free?
Setting the problem

- Lattice-theoretical properties of $P(E)$: make sense only in case $P(E)$ is a lattice (duh), that is, $E$ is square-free.
- Is there anything left in case $E$ is not square-free?
- It turns out that yes.
The extended permutohedron

What is it about?
An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

Getting past the “square-free” restriction

**Definition**

A subset $x$ of a transitive (binary) relation $e$ is closed if it is transitive, open if $e \setminus x$ is closed, regular closed if $x = \text{cl}(\text{int}(x))$, regular open if $x = \text{int}(\text{cl}(x))$, clopen if it is both open and closed. Operators $\text{cl}$ and $\text{int}$ defined as before: $\text{cl}(x)$ is the transitive closure of $x$, $\text{int}(x) = e \setminus \text{cl}(e \setminus x)$. 
Getting past the “square-free” restriction

Definition

A subset $x$ of a transitive (binary) relation $e$ is
Getting past the “square-free” restriction

Definition

A subset \( x \) of a transitive (binary) relation \( e \) is
- closed if it is transitive,
Getting past the “square-free” restriction

Definition

A subset $x$ of a transitive (binary) relation $e$ is
- closed if it is transitive,
- open if $e \setminus x$ is closed,
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

---

Getting past the “square-free” restriction

Definition

A subset $x$ of a transitive (binary) relation $e$ is
- **closed** if it is transitive,
- **open** if $e \setminus x$ is closed,
- **regular closed** if $x = \text{cl}(\text{int}(x))$,  

Operators $\text{cl}$ and $\text{int}$ defined as before: $\text{cl}(x)$ is the transitive closure of $x$,
$\text{int}(x) = e \setminus \text{cl}(e \setminus x)$.  

Getting past the “square-free” restriction

**Definition**

A subset $x$ of a **transitive** (binary) relation $e$ is
- **closed** if it is transitive,
- **open** if $e \setminus x$ is closed,
- **regular closed** if $x = \text{cl}(\text{int}(x))$,
- **regular open** if $x = \text{int}(\text{cl}(x))$. 
Getting past the “square-free” restriction

Definition

A subset $x$ of a transitive (binary) relation $e$ is

- **closed** if it is transitive,
- **open** if $e \setminus x$ is closed,
- **regular closed** if $x = \text{cl}(\text{int}(x))$,
- **regular open** if $x = \text{int}(\text{cl}(x))$,
- **clopen** if it is both open and closed.
Getting past the “square-free” restriction

Definition

A subset $x$ of a transitive (binary) relation $e$ is
- **closed** if it is transitive,
- **open** if $e \setminus x$ is closed,
- **regular closed** if $x = \text{cl}(\text{int}(x))$,
- **regular open** if $x = \text{int}(\text{cl}(x))$.
- **clopen** if it is both open and closed.

Operators $\text{cl}$ and $\text{int}$ defined as before: $\text{cl}(x)$ is the transitive closure of $x$, $\text{int}(x) = e \setminus \text{cl}(e \setminus x)$. 
The extended permutohedron

What is it about?
An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

The lattices $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$

Notation
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

The lattices $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$

**Notation**

For a transitive relation $e$, 

$x \mapsto x^c = e \setminus x$ defines a dual isomorphism between $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$.

$x \mapsto x^\perp = \text{cl}(x^c)$ defines an orthocomplementation on $\text{Reg}(e)$. 
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

The lattices $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$

Notation

For a transitive relation $e$,

$$\text{Clop}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is clopen} \}.$$  

$$\text{Reg}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is regular closed} \}.$$  

$$\text{Reg}_{\text{op}}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is regular open} \}.$$  

Let $x \mapsto \overline{x}$ define a dual isomorphism between $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$.

Let $x \mapsto \overline{x} = \text{cl}(\overline{x})$ define an orthocomplementation on $\text{Reg}(e)$. 
The extended permutohedron

What is it about?
An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

---

The lattices $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$

**Notation**

For a transitive relation $e$,

- $\text{Clop}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is clopen} \}$.
- $\text{Reg}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is regular closed} \}$.
- $\text{Reg}_{\text{op}}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is regular open} \}$.

- $x \mapsto x^c = e \setminus x$ defines a dual isomorphism between $\text{Reg}(e)$ and $\text{Reg}_{\text{op}}(e)$.
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in \( \text{Reg}(e) \)

Bip-Cambrian lattices

The lattices \( \text{Reg}(e) \) and \( \text{Reg}_{\text{op}}(e) \)

**Notation**

For a transitive relation \( e \),

\[
\text{Clop}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is clopen} \}.
\]

\[
\text{Reg}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is regular closed} \}.
\]

\[
\text{Reg}_{\text{op}}(e) \overset{\text{def.}}{=} \{ x \subseteq e \mid x \text{ is regular open} \}.
\]

- \( x \mapsto x^c = e \setminus x \) defines a dual isomorphism between \( \text{Reg}(e) \) and \( \text{Reg}_{\text{op}}(e) \).
- \( x \mapsto x^\perp = \text{cl}(x^c) \) defines an orthocomplementation on \( \text{Reg}(e) \).
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

The lattices (cont’d)

Proposition

Reg($e$) and $\text{Reg}^{\text{op}}(e)$ are isomorphic ortholattices, intersecting in $\text{Clop}(e)$.

$\text{Clop}(e)$ is an orthocomplemented poset.

It may not be a lattice (e.g., $\text{P}(E) = \text{Clop}(\delta E)$, for any poset $E$; take $E$ non-square-free).
Proposition

Reg(e) and Reg_{op}(e) are isomorphic ortholattices, intersecting in Clop(e).
The lattices (cont’d)

Proposition

Reg(e) and Reg_{\text{op}}(e) are isomorphic ortholattices, intersecting in Clop(e).

Clop(e) is an orthocomplemented poset.
The lattices (cont’d)

Proposition

Reg(e) and Reg_{op}(e) are isomorphic ortholattices, intersecting in Clop(e).

Clop(e) is an orthocomplemented poset. It may not be a lattice (e.g., P(E) = Clop(δ_E), for any poset E; take E non square-free).
Some notation

For a transitive relation $e$ on a set $E$, write

$$x \triangleleft e y \iff (x, y) \in e,$$
$$x \sqsubseteq e y \iff \text{(either } x \triangleleft e y \text{ or } x = y \text{)},$$

for all $x, y \in E$. We also set

$$[a, b]_e = \{ x \mid a \sqsubseteq e x \text{ and } x \sqsubseteq e b \},$$
$$[a, b[ e = \{ x \mid a \sqsubseteq e x \text{ and } x < e b \},$$
$$]a, b]_e = \{ x \mid a < e x \text{ and } x \sqsubseteq e b \},$$

for all $a, b \in E$. As $a \triangleleft e a$ may occur, $a$ may belong to $]a, b]_e$.
Square-free transitive relations

Definition

A transitive relation \( e \) on a set \( E \) is square-free if the preordered set \((E, \sqsubseteq e)\) is square-free. That is,\[
(\forall a, b, x, y)(a \sqsubseteq e x \text{ and } a \sqsubseteq e y \text{ and } x \sqsubseteq e b \text{ and } y \sqsubseteq e b \Rightarrow (either \ x \sqsubseteq e y \text{ or } y \sqsubseteq e x)).
\]
Square-free transitive relations

Definition

A transitive relation $e$ on a set $E$ is \textit{square-free} if the preordered set $(E, \trianglelefteq_e)$ is square-free. That is,

\[(\forall a, b, x, y)((a \trianglelefteq_e x \text{ and } a \trianglelefteq_e y \text{ and } x \trianglelefteq_e b \text{ and } y \trianglelefteq_e b) \implies (\text{either } x \trianglelefteq_e y \text{ or } y \trianglelefteq_e x)).\]
When is $\text{Clop}(e)$ a lattice?

**Theorem (Santocanale and W 2012)**

1. $\text{Clop}(e)$ is a lattice.
2. $\text{Clop}(e) = \text{Reg}(e)$.
3. $\text{int}(x)$ is closed, for any closed $x \subseteq e$.
4. $e$ is square-free.

The particular case where $e$ is antisymmetric is already taken care of by the abovementioned 1995 work by Pouzet, Reuter, Rival, and Zaguia. The particular case where $e$ is full (i.e., $e = E \times E$) follows from 2011 work by Hetyei and Krattenthaler. In that case, $e$ is always square-free, and $\text{Clop}(e) = \text{Reg}(e)$ is denoted by $\text{Bip}(E)$, the lattice of all bipartitions of a set $E$. 
When is $\text{Clop}(e)$ a lattice?

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any transitive relation $e$:

1. $\text{Clop}(e)$ is a lattice.
2. $\text{Clop}(e) = \text{Reg}(e)$.
3. $\text{int}(x)$ is closed, for any closed $x \subseteq e$.
4. $e$ is square-free.

The particular case where $e$ is antisymmetric is already taken care of by the abovementioned 1995 work by Pouzet, Reuter, Rival, and Zaguia. The particular case where $e$ is full (i.e., $e = E \times E$) follows from 2011 work by Hetyei and Krattenthaler. In that case, $e$ is always square-free, and $\text{Clop}(e) = \text{Reg}(e)$ is denoted by $\text{Bip}(E)$, the lattice of all bipartitions of a set $E$. 
When is Clop(\(e\)) a lattice?

Theorem (Santocanale and W 2012)

The following are equivalent, for any transitive relation \(e\):

1. Clop(\(e\)) is a lattice.
When is $\text{Clop}(e)$ a lattice?

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any transitive relation $e$:

1. $\text{Clop}(e)$ is a lattice.
2. $\text{Clop}(e) = \text{Reg}(e)$.

The particular case where $e$ is antisymmetric is already taken care of by the abovementioned 1995 work by Pouzet, Reuter, Rival, and Zaguia.

The particular case where $e$ is full (i.e., $e = E \times E$) follows from 2011 work by Hetyei and Krattenthaler. In that case, $e$ is always square-free, and $\text{Clop}(e) = \text{Reg}(e)$ is denoted by $\text{Bip}(E)$, the lattice of all bipartitions of a set $E$. 
When is \( \text{Clop}(e) \) a lattice?

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any transitive relation \( e \):

1. \( \text{Clop}(e) \) is a lattice.
2. \( \text{Clop}(e) = \text{Reg}(e) \).
3. \( \text{int}(x) \) is closed, for any closed \( x \subseteq e \).
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

When is $\text{Clop}(e)$ a lattice?

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any transitive relation $e$:

1. $\text{Clop}(e)$ is a lattice.
2. $\text{Clop}(e) = \text{Reg}(e)$.
3. $\text{int}(x)$ is closed, for any closed $x \subseteq e$.
4. $e$ is square-free.
The extended permutohedron

What is it about?
An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in \( \text{Reg}(e) \)

Bip-Cambrian lattices

Theorem (Santocanale and W 2012)

The following are equivalent, for any transitive relation \( e \):

1. \( \text{Clop}(e) \) is a lattice.
2. \( \text{Clop}(e) = \text{Reg}(e) \).
3. \( \text{int}(x) \) is closed, for any closed \( x \subseteq e \).
4. \( e \) is square-free.

The particular case where \( e \) is \textit{antisymmetric} is already taken care of by the abovementioned 1995 work by Pouzet, Reuter, Rival, and Zaguia.

The particular case where \( e \) is \textit{full} (i.e., \( e = E \times E \)) follows from 2011 work by Hetyei and Krattenthaler. In that case, \( e \) is always square-free, and \( \text{Clop}(e) = \text{Reg}(e) \) is denoted by \( \text{Bip}(E) \), the lattice of all bipartitions of a set \( E \).
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in Reg(e)
Bip-Cambrian lattices

When is Clop(e) a lattice?

Theorem (Santocanale and W 2012)
The following are equivalent, for any transitive relation e:

1. Clop(e) is a lattice.
2. Clop(e) = Reg(e).
3. int(x) is closed, for any closed x ⊆ e.
4. e is square-free.

- The particular case where e is antisymmetric is already taken care of by the abovementioned 1995 work by Pouzet, Reuter, Rival, and Zaguia.
- The particular case where e is full (i.e., e = E × E) follows from 2011 work by Hetyei and Krattenthaler.
When is $\text{Clop}(e)$ a lattice?

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any transitive relation $e$:

1. $\text{Clop}(e)$ is a lattice.
2. $\text{Clop}(e) = \text{Reg}(e)$.
3. $\text{int}(x)$ is closed, for any closed $x \subseteq e$.
4. $e$ is square-free.

- The particular case where $e$ is **antisymmetric** is already taken care of by the abovementioned 1995 work by Pouzet, Reuter, Rival, and Zaguia.
- The particular case where $e$ is **full** (i.e., $e = E \times E$) follows from 2011 work by Hetyei and Krattenthaler. In that case, $e$ is always square-free, and $\text{Clop}(e) = \text{Reg}(e)$ is denoted by $\text{Bip}(E)$, the lattice of all **bipartitions** of a set $E$. 
Recall that $P(E) = \text{Clop}(\delta_E)$, for any poset $E$. 
Permutohedra on non square-free posets

- Recall that $P(E) = \text{Clop}(\delta_E)$, for any poset $E$.
- Set $R(E) = \text{Reg}(\delta_E)$ (the extended permutohedron on $E$), for any poset $E$. 
The extended permutohedron

What is it about?
An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in Reg(e)

Bip-Cambrian lattices

Recall that $P(E) = \text{Clop}(\delta_E)$, for any poset $E$.

Set $R(E) = \text{Reg}(\delta_E)$ (the extended permutohedron on $E$), for any poset $E$.

In particular, $R(E)$ is always a lattice.
Permutohedra on non square-free posets

The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in Reg(e)
Bip-Cambrian lattices

Recall that $P(E) = \text{Clop}(\delta_E)$, for any poset $E$.

Set $R(E) = \text{Reg}(\delta_E)$ (the extended permutohedron on $E$), for any poset $E$.

In particular, $R(E)$ is always a lattice.

By earlier results, $P(E)$ is a lattice, iff $P(E) = R(E)$, iff $E$ is square-free.
The extended permutohedron on the square $B_2$

There it goes:

$$\begin{array}{ccc}
\text{B}_2 & 1 & \\ 0 & a & b \\
\end{array}$$

$$\begin{array}{ccc}
\text{R}(\text{B}_2) & a_0 & b_1 \\
 & c_{10} & c_{11} \\
 & c_{01} & c_{00} \\
 & b_0 & c_{11} \\
 & a_1 & a_1 \\
\end{array}$$

card $\text{R}(\text{B}_2)$ = 20 while card $\text{P}(\text{B}_2)$ = 18.

Every join-irreducible element of $\text{R}(\text{B}_2)$ is clopen (general explanation coming later).

The two elements $u$ and $u^\perp$ of $\text{R}(\text{B}_2)$ $\setminus \text{P}(\text{B}_2)$ are marked by doubled circles on the picture above.
The extended permutohedron on the square $B_2$

There it goes:

- card $R(B_2) = 20$ while card $P(B_2) = 18$. 
The extended permutohedron on the square $B_2$

There it goes:

- $\text{card } R(B_2) = 20$ while $\text{card } P(B_2) = 18$.
- Every join-irreducible element of $R(B_2)$ is clopen (general explanation coming later).
The extended permutohedron on the square $B_2$

There it goes:

- $\text{card } R(B_2) = 20$ while $\text{card } P(B_2) = 18$.
- Every join-irreducible element of $R(B_2)$ is clopen (general explanation coming later).
- The two elements $u$ and $u^\perp$ of $R(B_2) \setminus P(B_2)$ are marked by doubled circles on the picture above.
Basic observations

- Bip($n$) = Bip([n]) is the ortholattice of all binary relations $\mathbf{x}$ on [n] that are both transitive and co-transitive, ordered by $\subseteq$. 

---

The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in Reg(e)

Bip-Cambrian lattices
Basic observations

- $\text{Bip}(n) = \text{Bip}([n])$ is the ortholattice of all binary relations $\mathbf{x}$ on $[n]$ that are both \textit{transitive} and \textit{co-transitive}, ordered by $\subseteq$.

- The bipartition lattices $\text{Bip}(n)$ are “permutohedra without order”.

$\text{card Bip}(2) = 10$, $\text{card Bip}(3) = 74$, $\text{card Bip}(4) = 730$, $\text{card Bip}(5) = 9,002$.
Basic observations

- Bip\((n) = \text{Bip}([n])\) is the ortholattice of all binary relations \(x\) on \([n]\) that are both transitive and co-transitive, ordered by \(\subseteq\).
- The bipartition lattices Bip\((n)\) are “permutohedra without order”.
- \(\text{card } \text{Bip}(2) = 10, \text{card } \text{Bip}(3) = 74, \text{card } \text{Bip}(4) = 730, \text{card } \text{Bip}(5) = 9,002.\)
Basic observations

- \( \text{Bip}(n) = \text{Bip}([n]) \) is the ortholattice of all binary relations \( \mathbf{x} \) on \( [n] \) that are both transitive and co-transitive, ordered by \( \subseteq \).
- The bipartition lattices \( \text{Bip}(n) \) are “permutohedra without order”.
- \( \text{card Bip}(2) = 10 \), \( \text{card Bip}(3) = 74 \), \( \text{card Bip}(4) = 730 \), \( \text{card Bip}(5) = 9,002 \).
- Each \( \text{Bip}(n) \) is a graded lattice (Hetyei and Krattenthaler 2011).
Small bipartition lattices

Here is a picture of Bip(2), together with the join-dependency relation on its join-irreducible elements.

Bip(2)    The $D$ relation on Ji(Bip(2))
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

Small bipartition lattices

- Here is a picture of Bip(2), together with the join-dependency relation on its join-irreducible elements.

- In particular, Bip(2) is a bounded homomorphic image of a free lattice.

Bip(2)  The $D$ relation on $\text{Ji}(\text{Bip}(2))$
Small bipartition lattices

- Here is a picture of Bip(2), together with the join-dependency relation on its join-irreducible elements.

![Bip(2) Diagram]

- In particular, Bip(2) is a bounded homomorphic image of a free lattice.
- This does not extend to higher bipartition lattices: for example, M₃ embeds into Bip(3), so Bip(3) is not even semidistributive.
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

The lattice Bip(3)
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(\mathfrak{e})$
Bip-Cambrian lattices
Some open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice be embedded into some Bip(n)?


Is there a lattice (ortholattice) identity satisfied by every Bip(n)?
Some open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice be embedded into some Bip($n$)?
Some open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice be embedded into some Bip(n)?

Some open problems

Problem (Santocanale and W 2012)
Can every finite ortholattice be embedded into some Bip(n)?


Problem (Santocanale and W 2012)
Some open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice be embedded into some Bip($n$)?


Problem (Santocanale and W 2012)

Is there a lattice (ortholattice) identity satisfied by every Bip($n$)?
Some notation

- We denote by $C(e)$ the set of all triples $(a, b, U)$, where $(a, b) \in e$, $U \subseteq [a, b]_e$, and $a \neq b$ implies that $a \notin U$ and $b \in U$. 
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
 Completely join-irreducible elements in \( \text{Reg}(e) \)

Bip-Cambrian lattices

Some notation

- We denote by \( \mathcal{C}(e) \) the set of all triples \((a, b, U)\), where \((a, b) \in e\), \(U \subseteq [a, b]_e\), and \(a \neq b\) implies that \(a \not\in U\) and \(b \in U\).

- We set \( U^c = [a, b]_e \setminus U \), and

\[
\langle a, b; U \rangle = \begin{cases} 
\{(x, y) \mid a \triangleleft_e x \triangleleft_e y \triangleleft_e b, x \notin U, y \in U\} , & \text{if } a \neq b , \\
(\{a\} \cup U^c) \times (\{a\} \cup U) , & \text{if } a = b ,
\end{cases}
\]

for each \((a, b, U) \in \mathcal{C}(e)\).
Some notation

- We denote by $\mathcal{C}(e)$ the set of all triples $(a, b, U)$, where $(a, b) \in e$, $U \subseteq [a, b]_e$, and $a \neq b$ implies that $a \notin U$ and $b \in U$.

- We set $U^c = [a, b]_e \setminus U$, and

$$\langle a, b; U \rangle = \begin{cases} 
\{(x, y) \mid a \triangleleft_e x \triangleleft_e y \triangleleft_e b, x \notin U, y \in U\}, & \text{if } a \neq b, \\
({\{a\}} \cup U^c) \times ({\{a\}} \cup U), & \text{if } a = b,
\end{cases}$$

for each $(a, b, U) \in \mathcal{C}(e)$.

- Observe that $\langle a, b; U \rangle$ is bipartite (i.e., it cannot have both $(x, y)$ and $(y, z)$) iff $a \neq b$. If $a = b$, say that $\langle a, b; U \rangle$ is a clepsydra.
Recognizing the completely join-irreducible elements in $\text{Reg}(e)$

Theorem (Santocanale and W 2012)
Recognizing the completely join-irreducible elements in \( \text{Reg}(e) \)

**Theorem (Santocanale and W 2012)**

The following statements hold, for any transitive relation \( e \).
Recognizing the completely join-irreducible elements in $\text{Reg}(e)$

**Theorem (Santocanale and W 2012)**

The following statements hold, for any transitive relation $e$.

1. The completely join-irreducible elements of $\text{Reg}(e)$ are exactly the $\langle a, b; U \rangle$, where $(a, b, U) \in \mathcal{C}(e)$. **They are all clopen.**
Recognizing the completely join-irreducible elements in \( \text{Reg}(e) \)

**Theorem (Santocanale and W 2012)**

The following statements hold, for any transitive relation \( e \).

1. The completely join-irreducible elements of \( \text{Reg}(e) \) are exactly the \( \langle a, b; U \rangle \), where \( (a, b, U) \in \mathcal{C}(e) \). They are all clopen.

2. Every open (resp., regular closed) subset of \( e \) is a set-theoretical union (resp., join) of completely join-irreducible elements of \( \text{Reg}(e) \).
Recognizing the completely join-irreducible elements in $\text{Reg}(e)$

**Theorem (Santocanale and W 2012)**

The following statements hold, for any transitive relation $e$.

1. The completely join-irreducible elements of $\text{Reg}(e)$ are exactly the $\langle a, b; U \rangle$, where $(a, b, U) \in \mathcal{C}(e)$. They are all clopen.

2. Every open (resp., regular closed) subset of $e$ is a set-theoretical union (resp., join) of completely join-irreducible elements of $\text{Reg}(e)$.

**Corollary (Santocanale and W 2012)**
Recognizing the completely join-irreducible elements in $\text{Reg}(\mathbf{e})$

**Theorem (Santocanale and W 2012)**

The following statements hold, for any transitive relation $\mathbf{e}$.

1. The completely join-irreducible elements of $\text{Reg}(\mathbf{e})$ are exactly the $\langle a, b; U \rangle$, where $(a, b, U) \in \mathcal{C}(\mathbf{e})$. They are all clopen.

2. Every open (resp., regular closed) subset of $\mathbf{e}$ is a set-theoretical union (resp., join) of completely join-irreducible elements of $\text{Reg}(\mathbf{e})$.

**Corollary (Santocanale and W 2012)**

$\text{Reg}(\mathbf{e})$ is the Dedekind-MacNeille completion of $\text{Clop}(\mathbf{e})$, for any transitive relation $\mathbf{e}$.
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

The join-dependency relation on $\text{Reg}(e)$ in the antisymmetric case

**Lemma (Santocanale and W 2012)**

[Lemma content]

**Corollary (Santocanale and W 2012)**
The join-dependency relation on $\text{Reg}(e)$ is a strict ordering, for any finite, antisymmetric, transitive relation $e$.

**Corollary (Santocanale and W 2012)**
The lattice $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice, for any finite, antisymmetric, transitive relation $e$. 
The join-dependency relation on \( \text{Reg}(e) \) in the antisymmetric case

**Lemma (Santocanale and W 2012)**

Let \( e \) be a finite, antisymmetric, transitive relation and let \( p_i = \langle a_i, b_i; U_i \rangle \) be completely join-irreducible in \( \text{Reg}(e) \), for \( i \in \{0, 1\} \). Then \( p_0 \prec p_1 \) in \( \text{Reg}(e) \) iff \( [a_1, b_1]_e \not\subseteq [a_0, b_0]_e \) and \( U_1 = ((U_0 \cap [a_1, b_1]_e) \setminus \{a_1\}) \cup \{b_1\} \).
The join-dependency relation on $\text{Reg}(e)$ in the antisymmetric case

**Lemma (Santocanale and W 2012)**

Let $e$ be a finite, antisymmetric, transitive relation and let $p_i = \langle a_i, b_i; U_i \rangle$ be completely join-irreducible in $\text{Reg}(e)$, for $i \in \{0, 1\}$. Then $p_0 D p_1$ in $\text{Reg}(e)$ iff $[a_1, b_1]_e \subsetneq [a_0, b_0]_e$ and $U_1 = ((U_0 \cap [a_1, b_1]_e) \setminus \{a_1\}) \cup \{b_1\}$.

**Corollary (Santocanale and W 2012)**

The join-dependency relation on $\text{Reg}(e)$ is a strict ordering, for any finite, antisymmetric, transitive relation $e$.

The lattice $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice, for any finite, antisymmetric, transitive relation $e$. 
The join-dependency relation on $\text{Reg}(e)$ in the antisymmetric case

**Lemma (Santocanale and W 2012)**

Let $e$ be a finite, antisymmetric, transitive relation and let $p_i = \langle a_i, b_i; U_i \rangle$ be completely join-irreducible in $\text{Reg}(e)$, for $i \in \{0, 1\}$. Then $p_0 \leq_D p_1$ in $\text{Reg}(e)$ iff $[a_1, b_1]_e \not\subseteq [a_0, b_0]_e$ and $U_1 = ((U_0 \cap [a_1, b_1]_e) \setminus \{a_1\}) \cup \{b_1\}$.

**Corollary (Santocanale and W 2012)**

The join-dependency relation on $\text{Reg}(e)$ is a strict ordering, for any finite, antisymmetric, transitive relation $e$. 
The join-dependency relation on $\text{Reg}(e)$ in the antisymmetric case

**Lemma (Santocanale and W 2012)**

Let $e$ be a finite, antisymmetric, transitive relation and let $p_i = \langle a_i, b_i; U_i \rangle$ be completely join-irreducible in $\text{Reg}(e)$, for $i \in \{0, 1\}$. Then $p_0 D p_1$ in $\text{Reg}(e)$ iff $[a_1, b_1]_e \subsetneq [a_0, b_0]_e$ and $U_1 = ((U_0 \cap [a_1, b_1]_e) \setminus \{a_1\}) \cup \{b_1\}$.

**Corollary (Santocanale and W 2012)**

The join-dependency relation on $\text{Reg}(e)$ is a strict ordering, for any finite, antisymmetric, transitive relation $e$.

**Corollary (Santocanale and W 2012)**

The lattice $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice, for any finite, antisymmetric, transitive relation $e$. 
The join-dependency relation on $\text{Reg}(e)$ in the antisymmetric case

**Lemma (Santocanale and W 2012)**

Let $e$ be a finite, antisymmetric, transitive relation and let $p_i = \langle a_i, b_i; U_i \rangle$ be completely join-irreducible in $\text{Reg}(e)$, for $i \in \{0, 1\}$. Then $p_0 D p_1$ in $\text{Reg}(e)$ iff $[a_1, b_1]_e \not\subseteq [a_0, b_0]_e$ and $U_1 = ((U_0 \cap [a_1, b_1]_e) \setminus \{a_1\}) \cup \{b_1\}$.

**Corollary (Santocanale and W 2012)**

The join-dependency relation on $\text{Reg}(e)$ is a strict ordering, for any finite, antisymmetric, transitive relation $e$.

**Corollary (Santocanale and W 2012)**

The lattice $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice, for any finite, antisymmetric, transitive relation $e$. 
Bounded lattices $\text{Reg}(e)$

In particular, $R(E)$ is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset $E$. 
Bounded lattices $\text{Reg}(e)$

In particular, $R(E)$ is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset $E$.

**Theorem (Santocanale and W 2012)**
Bounded lattices $\text{Reg}(e)$

In particular, $R(E)$ is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset $E$.

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any finite, transitive relation $e$:

1. $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice.
2. $\text{Reg}(e)$ is semidistributive.
3. $\text{Reg}(e)$ is pseudocomplemented.
4. Every connected component of the preordering $\sqsubseteq e$ is either antisymmetric or has the form $\{a, b\}$ with $a \neq b$, $(a, b) \in e$, and $(b, a) \in e$.

Hence if $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice, then it is a direct product of extended permutohedra on finite posets and copies of $\{0, 1\}$ and $\text{Bip}(2)$.
Bounded lattices Reg(e)

In particular, R(E) is a bounded homomorphic image of a free lattice, for any finite \((\text{not necessarily square-free})\) poset \(E\).

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any finite, transitive relation \(e\):

1. \(\text{Reg}(e)\) is a bounded homomorphic image of a free lattice.
Bounded lattices $\text{Reg}(e)$

In particular, $R(E)$ is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset $E$.

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any finite, transitive relation $e$:

1. $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice.
2. $\text{Reg}(e)$ is semidistributive.
Bounded lattices $\text{Reg}(e)$

In particular, $R(E)$ is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset $E$.

Theorem (Santocanale and W 2012)

The following are equivalent, for any finite, transitive relation $e$:

1. $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice.
2. $\text{Reg}(e)$ is semidistributive.
3. $\text{Reg}(e)$ is pseudocomplemented.
Bounded lattices $\text{Reg}(e)$

In particular, $R(E)$ is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset $E$.

**Theorem (Santocanale and W 2012)**

The following are equivalent, for any finite, transitive relation $e$:

1. $\text{Reg}(e)$ is a bounded homomorphic image of a free lattice.
2. $\text{Reg}(e)$ is semidistributive.
3. $\text{Reg}(e)$ is pseudocomplemented.
4. Every connected component of the preordering $\preceq_e$ is either antisymmetric or has the form $\{a, b\}$ with $a \neq b$, $(a, b) \in e$, and $(b, a) \in e$. 
Bounded lattices \( \text{Reg}(e) \)

In particular, \( R(E) \) is a bounded homomorphic image of a free lattice, for any finite (not necessarily square-free) poset \( E \).

Theorem (Santocanale and W 2012)

The following are equivalent, for any finite, transitive relation \( e \):

1. \( \text{Reg}(e) \) is a bounded homomorphic image of a free lattice.
2. \( \text{Reg}(e) \) is semidistributive.
3. \( \text{Reg}(e) \) is pseudocomplemented.
4. Every connected component of the preordering \( \preceq_e \) is either antisymmetric or has the form \( \{a, b\} \) with \( a \neq b \), \( (a, b) \in e \), and \( (b, a) \in e \).

Hence if \( \text{Reg}(e) \) is a bounded homomorphic image of a free lattice, then it is a direct product of extended permutohedra on finite posets and copies of \( \{0, 1\} \) and \( \text{Bip}(2) \).
More open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into $\text{Reg}(E)$, for some finite poset $E$?

Problem (Santocanale and W 2012)

Is there a nontrivial ortholattice identity that holds in $\text{Reg}(E)$ for any finite poset $E$?

Not even known for $E$ a finite chain:

Problem (Santocanale and W 2011)

Is there a nontrivial lattice (ortholattice) identity that holds in $P(n)$ for any positive integer $n$?
More open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into $R(E)$, for some finite poset $E$?
More open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into $R(E)$, for some finite poset $E$?

Problem (Santocanale and W 2012)

Is there a nontrivial ortholattice identity that holds in $R(E)$ for any finite poset $E$?

Not even known for $E$ a finite chain:

Problem (Santocanale and W 2011)

Is there a nontrivial lattice (ortholattice) identity that holds in $P(n)$ for any positive integer $n$?
More open problems

Problem (Santocanale and W 2012)

Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into $R(E)$, for some finite poset $E$?

Problem (Santocanale and W 2012)

Is there a nontrivial ortholattice identity that holds in $R(E)$ for any finite poset $E$?
More open problems

**Problem (Santocanale and W 2012)**

Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into \( R(E) \), for some finite poset \( E \)?

**Problem (Santocanale and W 2012)**

Is there a nontrivial ortholattice identity that holds in \( R(E) \) for any finite poset \( E \)?

Not even known for \( E \) a finite chain:
More open problems

Problem (Santocanale and W 2012)
Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into $R(E)$, for some finite poset $E$?

Problem (Santocanale and W 2012)
Is there a nontrivial ortholattice identity that holds in $R(E)$ for any finite poset $E$?

Not even known for $E$ a finite chain:

Problem (Santocanale and W 2011)
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in Reg(\(e\))
Bip-Cambrian lattices

More open problems

Problem (Santocanale and W 2012)
Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into \(R(E)\), for some finite poset \(E\)?

Problem (Santocanale and W 2012)
Is there a nontrivial ortholattice identity that holds in \(R(E)\) for any finite poset \(E\)?

Not even known for \(E\) a finite chain:

Problem (Santocanale and W 2011)
Is there a nontrivial lattice (ortholattice) identity that holds in \(P(n)\) for any positive integer \(n\)?
Minimal subdirect decomposition of the permutohedron $P(n)$

- For $U \subseteq [n]$, denote by $P_U(n)$ the set of all transitive $x \in \mathcal{J}_n$ such that
Minimal subdirect decomposition of the permutohedron $P(n)$

- For $U \subseteq [n]$, denote by $P_U(n)$ the set of all transitive $x \in J_n$ such that

\[(i < j < k \text{ and } (i, k) \in x) \Rightarrow \begin{cases} (i, j) \in x & \text{(if } j \in U), \\ (j, k) \in x & \text{(if } j \notin U). \end{cases}\]

- $P_U(n)$ is a sublattice of $P(n)$. More is true: 

$P(\emptyset)(n) \cong P([n])$ is the Tamari lattice on $n + 1$ letters (associahedron).
Minimal subdirect decomposition of the permutohedron $P(n)$

- For $U \subseteq [n]$, denote by $P_U(n)$ the set of all transitive $x \in J_n$ such that

$$ (i < j < k \text{ and } (i, k) \in x) \Rightarrow \begin{cases} (i, j) \in x & \text{(if } j \in U), \\ (j, k) \in x & \text{(if } j \notin U). \end{cases} $$

- $P_U(n)$ is a sublattice of $P(n)$. More is true:

**Theorem (Santocanale and W 2011)**

Each $P_U(n)$ is a lattice-theoretical retract of $P(n)$, and $P(n)$ is a subdirect product of all $P_U(n)$. Furthermore, the $P_U(n)$ are isomorphic to Reading’s Cambrian lattices of type $A$.

$P_{\emptyset}(n) \simeq P_{[n]}(n)$ is the Tamari lattice on $n + 1$ letters (associahedron).
Minimal subdirect decomposition of the permutohedron $P(n)$

- For $U \subseteq [n]$, denote by $P_U(n)$ the set of all transitive $x \in \mathcal{J}_n$ such that
  
  $$(i < j < k \text{ and } (i, k) \in x) \Rightarrow \begin{cases} (i, j) \in x & (\text{if } j \in U), \\ (j, k) \in x & (\text{if } j \notin U). \end{cases}$$

- $P_U(n)$ is a sublattice of $P(n)$. More is true:

**Theorem (Santocanale and W 2011)**

Each $P_U(n)$ is a lattice-theoretical retract of $P(n)$, and $P(n)$ is a subdirect product of all $P_U(n)$. 
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

Minimal subdirect decomposition of the permutohedron $P(n)$

- For $U \subseteq [n]$, denote by $P_U(n)$ the set of all transitive $x \in J_n$ such that

$$(i < j < k \text{ and } (i, k) \in x) \Rightarrow \begin{cases} (i, j) \in x & \text{(if } j \in U), \\ (j, k) \in x & \text{(if } j \notin U). \end{cases}$$

- $P_U(n)$ is a sublattice of $P(n)$. More is true:

Theorem (Santocanale and W 2011)

Each $P_U(n)$ is a lattice-theoretical retract of $P(n)$, and $P(n)$ is a subdirect product of all $P_U(n)$. Furthermore, the $P_U(n)$ are isomorphic to N. Reading’s Cambrian lattices of type A.
Minimal subdirect decomposition of the permutohedron $P(n)$

- For $U \subseteq [n]$, denote by $P_U(n)$ the set of all transitive $x \in J_n$ such that

  $$(i < j < k \text{ and } (i, k) \in x) \Rightarrow \begin{cases} (i, j) \in x & \text{(if } j \in U), \\ (j, k) \in x & \text{(if } j \notin U). \end{cases}$$

- $P_U(n)$ is a sublattice of $P(n)$. More is true:

**Theorem (Santocanale and W 2011)**

Each $P_U(n)$ is a lattice-theoretical retract of $P(n)$, and $P(n)$ is a subdirect product of all $P_U(n)$. Furthermore, the $P_U(n)$ are isomorphic to N. Reading’s Cambrian lattices of type A.

$P_\emptyset(n) \cong P_{[n]}(n)$ is the Tamari lattice on $n + 1$ letters (associahedron).
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(e)$

Bip-Cambrian lattices

Picturing the Cambrian lattices of type A, for $n = 4$
N. Reading observed that each $P_U(n)$ has cardinality $\frac{1}{n+1}\binom{2n}{n}$.
Minimal subdirect decomposition of Bip($n$)

- $a \in [n]$ is isolated in $x \in \text{Bip}(n)$ if $((i, a) \in x$ and $(a, i) \in x) \iff i = a$, $\forall i \in [n]$. 

The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices
Minimal subdirect decomposition of Bip($n$)

- $a \in [n]$ is isolated in $x \in \text{Bip}(n)$ if $((i, a) \in x$ and $(a, i) \in x) \iff i = a$, $\forall i \in [n]$.

- For $0 \leq k < n$, $a \in [n]$, and $U \subseteq [n] \setminus \{a\}$ with $k$ elements, denote $(\ldots)$ by $S(n, k)$ the poset of all $x \in \text{Bip}(n)$ such that each isolated point of $x$ is equal to $a$, and if $a$ is isolated, then $(U^c \times \{a\}) \cup (\{a\} \times U) \subseteq x$. 

Theorem (Santocanale and W 2012) Bip($n$) is a subdirect product of copies of the $S(n, k)$ (minimal subdirect decomposition).
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

Minimal subdirect decomposition of Bip($n$)

- $a \in [n]$ is isolated in $x \in \text{Bip}(n)$ if $((i, a) \in x$ and $(a, i) \in x) \iff i = a$, $\forall i \in [n]$.

- For $0 \leq k < n$, $a \in [n]$, and $U \subseteq [n] \setminus \{a\}$ with $k$ elements, denote $(\ldots)$ by $S(n, k)$ the poset of all $x \in \text{Bip}(n)$ such that each isolated point of $x$ is equal to $a$, and if $a$ is isolated, then $(U^c \times \{a\}) \cup (\{a\} \times U) \subseteq x$.

- $S(n, k)$ is a self-dual lattice (not necessarily a sublattice of Bip($n$)), and $S(n, k) \cong S(n, n - 1 - k)$ (so it suffices to consider $0 \leq 2k < n$).
The extended permutohedron

What is it about?

An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in $\text{Reg}(\mathfrak{e})$

Bip-Cambrian lattices

---

Minimal subdirect decomposition of $\text{Bip}(n)$

- $a \in [n]$ is isolated in $x \in \text{Bip}(n)$ if $((i, a) \in x$ and $(a, i) \in x) \iff i = a$, $\forall i \in [n]$.

- For $0 \leq k < n$, $a \in [n]$, and $U \subseteq [n] \setminus \{a\}$ with $k$ elements, denote $(\ldots)$ by $S(n, k)$ the poset of all $x \in \text{Bip}(n)$ such that each isolated point of $x$ is equal to $a$, and if $a$ is isolated, then $(U^c \times \{a\}) \cup (\{a\} \times U) \subseteq x$.

- $S(n, k)$ is a self-dual lattice (not necessarily a sublattice of $\text{Bip}(n)$), and $S(n, k) \cong S(n, n - 1 - k)$ (so it suffices to consider $0 \leq 2k < n$).

---

Theorem (Santocanale and W 2012)
Minimal subdirect decomposition of Bip($n$)

- $a \in [n]$ is isolated in $x \in \text{Bip}(n)$ if $((i, a) \in x$ and $(a, i) \in x) \iff i = a$, $\forall i \in [n]$.

- For $0 \leq k < n$, $a \in [n]$, and $U \subseteq [n] \setminus \{a\}$ with $k$ elements, denote $(\ldots)$ by $S(n, k)$ the poset of all $x \in \text{Bip}(n)$ such that each isolated point of $x$ is equal to $a$, and if $a$ is isolated, then $(U^c \times \{a\}) \cup (\{a\} \times U) \subseteq x$.

- $S(n, k)$ is a self-dual lattice (not necessarily a sublattice of Bip($n$)), and $S(n, k) \cong S(n, n - 1 - k)$ (so it suffices to consider $0 \leq 2k < n$).

Theorem (Santocanale and W 2012)

Bip($n$) is a subdirect product of copies of the $S(n, k)$ (minimal subdirect decomposition).
The extended permutohedron

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in Reg(e)
Bip-Cambrian lattices

Cardinalities for small values: card $S(3, 0) = 24$, card $S(3, 1) = 21$; card $S(4, 0) = 158$, card $S(4, 1) = 142$; card $S(5, 0) = 1,320$, card $S(5, 1) = 1,202$, card $S(5, 2) = 1,198$. 

Hence $\text{card } S(n, k)$ depends on $k$. 

Recall the picture of Bip(3):
The bip-Cambrian lattices $S(n, k)$

- Cardinalities for small values: $\text{card } S(3, 0) = 24$, $\text{card } S(3, 1) = 21$; $\text{card } S(4, 0) = 158$, $\text{card } S(4, 1) = 142$; $\text{card } S(5, 0) = 1,320$, $\text{card } S(5, 1) = 1,202$, $\text{card } S(5, 2) = 1,198$. Hence $\text{card } S(n, k)$ depends on $k$. 

Recall the picture of $\text{Bip}(3)$:
The bip-Cambrian lattices $S(n, k)$

- Cardinalities for small values: $\text{card } S(3, 0) = 24$, $\text{card } S(3, 1) = 21$; $\text{card } S(4, 0) = 158$, $\text{card } S(4, 1) = 142$; $\text{card } S(5, 0) = 1,320$, $\text{card } S(5, 1) = 1,202$, $\text{card } S(5, 2) = 1,198$. Hence $\text{card } S(n, k)$ depends on $k$.
- Recall the picture of Bip(3):

![Diagram of Bip(3)]
The extended permutohedron

What is it about?
An extension to every poset

Regular closed subsets of a transitive relation

Back to bipartitions

Completely join-irreducible elements in Reg(\(e\))

Bip-Cambrian lattices

Pictures of S(3, 0) and S(3, 1)
The congruence lattice of Bip($n$)

The description of all join-irreducible elements of Bip($n$) (and their $D$ relation) makes it possible to prove the following.
The congruence lattice of Bip($n$)

The description of all join-irreducible elements of Bip($n$) (and their $D$ relation) makes it possible to prove the following.

**Lemma (Santocanale and W 2012)**
The congruence lattice of Bip($n$)

The description of all join-irreducible elements of Bip($n$) (and their $D$ relation) makes it possible to prove the following.

**Lemma (Santocanale and W 2012)**

Let $p$ and $q$ be join-irreducible elements in Bip($n$), where $n \geq 3$. Then $\text{con}(p_*, p) \subseteq \text{con}(q_*, q)$ iff either $q$ is bipartite or $p = q$ is a clepsydra.
The extended permutohedral

What is it about?
An extension to every poset
Regular closed subsets of a transitive relation
Back to bipartitions
Completely join-irreducible elements in $\text{Reg}(e)$
Bip-Cambrian lattices

The description of all join-irreducible elements of $\text{Bip}(n)$ (and their $D$ relation) makes it possible to prove the following.

**Lemma (Santocanale and W 2012)**
Let $p$ and $q$ be join-irreducible elements in $\text{Bip}(n)$, where $n \geq 3$. Then $\text{con}(p_*, p) \subseteq \text{con}(q_*, q)$ iff either $q$ is bipartite or $p = q$ is a clepsydra.

**Corollary (Santocanale and W 2012)**
Let $n \geq 3$. Then the congruence lattice of $\text{Bip}(n)$ is Boolean on $n \cdot 2^{n-1}$ atoms, with a top element added.