Large lower finite lattices with breadth three

Friedrich Wehrung

Université de Caen
LMNO, UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: wehrung@math.unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung

ROGICS’08
A lattice $L$ with zero is **lower finite**, if

$L \downarrow a := \{ x \in L \mid x \leq a \}$ is finite for each $a \in L$. 
A lattice $L$ with zero is **lower finite**, if $L \downarrow a := \{ x \in L \mid x \leq a \}$ is finite for each $a \in L$.

We say that $L$ is a **$k$-ladder**, if it is lower finite and every element of $L$ has at most $k$ lower covers.
A lattice $L$ with zero is lower finite, if $L \downarrow a := \{ x \in L \mid x \leq a \}$ is finite for each $a \in L$.

We say that $L$ is a $k$-ladder, if it is lower finite and every element of $L$ has at most $k$ lower covers.

We say that $L$ has breadth $\leq k$, if for every nonempty finite $X \subseteq L$, there exists $Y \subseteq X$ such that $|Y| \leq k$ and $\bigvee X = \bigvee Y$. 
A simple relation between ladders and breadth

Lemma

Every $k$-ladder has breadth $\leq k$. 
A simple relation between ladders and breadth

Lemma

Every $k$-ladder has breadth $\leq k$.
The converse is false.
A simple relation between ladders and breadth

**Lemma**

Every $k$-ladder has breadth $\leq k$. The converse is false.

The lattice $M_3$ below has breadth 2. It is a 3-ladder but not a 2-ladder.
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set

$[\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \};$

Kuratowski's Free Set Theorem (1951)
Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi: [\Omega]^k \to [\Omega]^{<\omega}$, there exists $H \in [\Omega]^{k+1}$ such that $x/ \in \Phi(H \{x\})$ for each $x \in H$.

(We say that $H$ is free with respect to $\Phi$.)

For a $k$-ladder (or even a lattice of breadth $\leq k$), $L$, we obtain, by applying this to the map $X \mapsto \downarrow \bigvee X$,

Proposition (S. Z. Ditor, 1984)
Let $k$ be a positive integer. Then every lower finite lattice $L$ of breadth $\leq k$ has cardinality at most $\aleph_k - 1$. 

3-ladders
Background
Critical points
MA($\aleph_1$; precaliber $\aleph_1$)
Morasses
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set

- $[\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \}$;
- $[\Omega]^{<\omega} := \{ X \subseteq \Omega \mid X \text{ is finite} \}$.
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set

- $[\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \}$;
- $[\Omega]<\omega := \{ X \subseteq \Omega \mid X \text{ is finite} \}$.

Kuratowski’s Free Set Theorem (1951)
For any set $\Omega$ and any positive integer $n$, we set
- $[\Omega]^n := \{X \subseteq \Omega \mid |X| = n\}$;
- $[\Omega]^{<\omega} := \{X \subseteq \Omega \mid X \text{ is finite}\}$.

**Kuratowski’s Free Set Theorem (1951)**

Let $k$ be a positive integer and let $\Omega$ be a set.
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set
- $[\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \}$;
- $[\Omega]^{<\omega} := \{ X \subseteq \Omega \mid X \text{ is finite} \}$.

Kuratowski’s Free Set Theorem (1951)

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff

\[
\text{for every } \Phi: [\Omega]^k \to [\Omega]^{<\omega}, \text{ there exists } H \in [\Omega]^{k+1} \text{ such that } x /\in \Phi(H \setminus \{x\}) \text{ for each } x \in H.
\]
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set
- $[\Omega]^n := \{X \subseteq \Omega \mid |X| = n\}$;
- $[\Omega]^{<\omega} := \{X \subseteq \Omega \mid X \text{ is finite}\}$.

Kuratowski’s Free Set Theorem (1951)

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi : [\Omega]^k \to [\Omega]^{<\omega}$,
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set
- $[\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \}$;
- $[\Omega]^{<\omega} := \{ X \subseteq \Omega \mid X \text{ is finite} \}$.

**Kuratowski’s Free Set Theorem (1951)**

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi : [\Omega]^k \to [\Omega]^{<\omega}$, there exists $H \in [\Omega]^{k+1}$ such that
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set

- $[\Omega]^n := \{X \subseteq \Omega \mid |X| = n\}$;
- $[\Omega]^{<\omega} := \{X \subseteq \Omega \mid X \text{ is finite}\}$.

**Kuratowski’s Free Set Theorem (1951)**

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi: [\Omega]^k \to [\Omega]^{<\omega}$, there exists $H \in [\Omega]^{k+1}$ such that $x \notin \Phi(H \setminus \{x\})$ for each $x \in H$. 
An upper bound for the size of a \( k \)-ladder

For any set \( \Omega \) and any positive integer \( n \), we set

- \([\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \}\);  
- \([\Omega]<\omega := \{ X \subseteq \Omega \mid X \text{ is finite} \}\).

**Kuratowski’s Free Set Theorem (1951)**

Let \( k \) be a positive integer and let \( \Omega \) be a set. Then \( |\Omega| \geq \aleph_k \) iff for every \( \Phi : [\Omega]^k \rightarrow [\Omega]<\omega \), there exists \( H \in [\Omega]^{k+1} \) such that \( x \notin \Phi(H \setminus \{x\}) \) for each \( x \in H \). (*We say that \( H \) is free with respect to \( \Phi \).*).
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set
- $[\Omega]^n := \{X \subseteq \Omega \mid |X| = n\}$;
- $[\Omega]^{<\omega} := \{X \subseteq \Omega \mid X$ is finite$\}$.

**Kuratowski’s Free Set Theorem (1951)**

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi : [\Omega]^k \rightarrow [\Omega]^{<\omega}$, there exists $H \in [\Omega]^{k+1}$ such that $x \notin \Phi(H \setminus \{x\})$ for each $x \in H$. *(We say that $H$ is free with respect to $\Phi$).*

For a $k$-ladder (or even a lattice of breadth $\leq k$) $L$,
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set

- $[\Omega]^n := \{ X \subseteq \Omega \mid |X| = n \}$;
- $[\Omega]^<\omega := \{ X \subseteq \Omega \mid X \text{ is finite} \}$.

Kuratowski’s Free Set Theorem (1951)

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi: [\Omega]^k \rightarrow [\Omega]^<\omega$, there exists $H \in [\Omega]^{k+1}$ such that $x \notin \Phi(H \setminus \{x\})$ for each $x \in H$. (We say that $H$ is free with respect to $\Phi$.)

For a $k$-ladder (or even a lattice of breadth $\leq k$) $L$, we obtain, by applying this to the map $X \mapsto L \downarrow \bigvee X$, ...
An upper bound for the size of a $k$-ladder

For any set $\Omega$ and any positive integer $n$, we set

- $[\Omega]^n := \{X \subseteq \Omega \mid |X| = n\}$;
- $[\Omega]<\omega := \{X \subseteq \Omega \mid X \text{ is finite}\}$.

**Kuratowski’s Free Set Theorem (1951)**

Let $k$ be a positive integer and let $\Omega$ be a set. Then $|\Omega| \geq \aleph_k$ iff for every $\Phi: [\Omega]^k \rightarrow [\Omega]<\omega$, there exists $H \in [\Omega]^k+1$ such that $x \notin \Phi(H \setminus \{x\})$ for each $x \in H$. (We say that $H$ is free with respect to $\Phi$.)

For a $k$-ladder (or even a lattice of breadth $\leq k$) $L$, we obtain, by applying this to the map $X \mapsto L \downarrow \bigvee X$,

**Proposition (S. Z. Ditor, 1984)**

Let $k$ be a positive integer. Then every lower finite lattice $L$ of breadth $\leq k$ has cardinality at most $\aleph_{k-1}$.
1-ladders and 2-ladders

- Every finite chain is a 1-ladder.
1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers.
1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers. There are no other 1-ladders.
1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers. There are no other 1-ladders.
- And 2-ladders?
1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers. There are no other 1-ladders.
- And 2-ladders?

**Theorem (S. Z. Ditor 1984)**

There exists a 2-ladder of cardinality $\aleph_1$. 

1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers. There are no other 1-ladders.
- And 2-ladders?

**Theorem (S. Z. Ditor 1984)**

There exists a 2-ladder of cardinality $\aleph_1$.

**Examples of applications:**

Every distributive algebraic lattice with $\leq \aleph_1$ compact elements is isomorphic to
1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers. There are no other 1-ladders.
- And 2-ladders?

**Theorem (S. Z. Ditor 1984)**

There exists a 2-ladder of cardinality $\aleph_1$.

**Examples of applications:**

Every distributive algebraic lattice with $\leq \aleph_1$ compact elements is isomorphic to
- the congruence lattice of some lattice (A. P. Huhn 1989).
1-ladders and 2-ladders

- Every finite chain is a 1-ladder. So is the chain $\omega$ of all natural numbers. There are no other 1-ladders.
- And 2-ladders?

**Theorem (S. Z. Ditor 1984)**

There exists a 2-ladder of cardinality $\aleph_1$.

**Examples of applications:**

Every distributive algebraic lattice with $\leq \aleph_1$ compact elements is isomorphic to

- the congruence lattice of some lattice (A. P. Huhn 1989).
- the lattice of all normal subgroups of some locally finite group (P. Růžička, J. Tůma, and F. Wehrung 2006).
Proof of existence of a 2-ladder of cardinality $\aleph_1$: 

We construct $F := \bigcup (F_\alpha | \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. At limit stages $\lambda < \omega_1$, set $F_\lambda := \bigcup (F_\alpha | \alpha < \lambda)$. The problem is the successor case. Suppose $F_\alpha$ constructed. It is a countable 2-ladder (induction hypothesis).
2-ladders (continued)

**Proof of existence of a 2-ladder of cardinality $\aleph_1$:** We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$,
Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively.
2-ladders (continued)

Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. 
Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. At limit stages $\lambda < \omega_1$,
2-ladders (continued)

Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. At limit stages $\lambda < \omega_1$, set $F_\lambda := \bigcup (F_\alpha \mid \alpha < \lambda)$. 
Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. At limit stages $\lambda < \omega_1$, set $F_\lambda := \bigcup (F_\alpha \mid \alpha < \lambda)$. The problem is the successor case.
Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. At limit stages $\lambda < \omega_1$, set $F_\lambda := \bigcup (F_\alpha \mid \alpha < \lambda)$. The problem is the successor case. Suppose $F_\alpha$ constructed.
Proof of existence of a 2-ladder of cardinality $\aleph_1$: We construct $F := \bigcup (F_\alpha \mid \alpha < \omega_1)$, the $F_\alpha$s constructed inductively. Start with $F_0 := \{0\}$. At limit stages $\lambda < \omega_1$, set $F_\lambda := \bigcup (F_\alpha \mid \alpha < \lambda)$. The problem is the successor case. Suppose $F_\alpha$ constructed. It is a countable 2-ladder (induction hypothesis).
2-ladders (continued further)
Pick a cofinal chain $C$ of $F_\alpha$. 
2-ladders (continued further)

Pick a cofinal chain $C$ of $F_\alpha$.
Add a copy $C' = \{x' \mid x \in C\} \cong C \ldots$
2-ladders (continued further)

Pick a cofinal chain $C$ of $F_\alpha$.
Add a copy $C' = \{x' \mid x \in C\} \cong C$.
And we are done ($F_{\alpha+1} := F_\alpha \cup C'$)!
Critical points: basic definitions

- Denote by $\text{Con}_c A$ the $(\lor, 0)$-semilattice of all compact (i.e., finitely generated) congruences of an algebra $A$. 
Critical points: basic definitions

- Denote by $\text{Con}_c A$ the $(\vee, 0)$-semilattice of all compact (i.e., finitely generated) congruences of an algebra $A$.
- For a class $\mathcal{C}$ of algebras, put

$$\text{Con}_c \mathcal{C} := \{ S \mid (\exists A \in \mathcal{C})(S \cong \text{Con}_c A) \}.$$
Critical points: basic definitions

- Denote by $\text{Con}_c A$ the $(\vee, 0)$-semilattice of all compact (i.e., finitely generated) congruences of an algebra $A$.

- For a class $C$ of algebras, put

  $$\text{Con}_c C := \{ S \mid (\exists A \in C)(S \cong \text{Con}_c A) \}.$$ 

- For classes $\mathcal{A}$ and $\mathcal{B}$ of algebras, denote by $\text{crit}(\mathcal{A}, \mathcal{B})$ (critical point of $(\mathcal{A}, \mathcal{B})$) the least possible value of $|S|$ where $S \in \text{Con}_c \mathcal{A} \setminus \text{Con}_c \mathcal{B}$, if it exists; $\infty$, otherwise (i.e., if $\text{Con}_c \mathcal{A} \subseteq \text{Con}_c \mathcal{B}$).
Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:
Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_0$;

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_1$;

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_2$. 
Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_n$, for some natural number $n$;
Critical points (continued)

Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_n$, for some natural number $n$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \infty$. 

Finitely generated varieties $\mathcal{A}$ and $\mathcal{B}$ of (bounded) lattices have been found with either one of the following situations:

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_0$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_1$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_2$. 

Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_n$, for some natural number $n$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \infty$.

Finitely generated varieties $\mathcal{A}$ and $\mathcal{B}$ of (bounded) lattices have been found with either one of the following situations:


**Theorem (P. Gillibert 2007)**

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_n$, for some natural number $n$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \infty$.

Finitely generated varieties $\mathcal{A}$ and $\mathcal{B}$ of (bounded) lattices have been found with either one of the following situations:

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_0$;
Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_n$, for some natural number $n$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \infty$.

Finitely generated varieties $\mathcal{A}$ and $\mathcal{B}$ of (bounded) lattices have been found with either one of the following situations:

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_0$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_1$;
Theorem (P. Gillibert 2007)

For every locally finite variety $\mathcal{A}$ and every finitely generated congruence-distributive variety $\mathcal{B}$, exactly one of the following holds:

- $\text{crit}(\mathcal{A}, \mathcal{B})$ is finite;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_n$, for some natural number $n$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \infty$.

Finitely generated varieties $\mathcal{A}$ and $\mathcal{B}$ of (bounded) lattices have been found with either one of the following situations:

- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_0$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_1$;
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_2$. 
More specifically, 

- \( \text{crit}(\mathcal{M}_3^{01}, \mathcal{D}^{01}) = \aleph_0 \) and \( \text{crit}(\mathcal{M}_4^{01}, \mathcal{M}_3^{01}) = \aleph_2 \) (M. Ploščica 2000, 2003) (later extended to unbounded lattices by P. Gillibert);
More specifically,

- $\text{crit} (\mathcal{M}_3^{01}, \mathcal{D}^{01}) = \aleph_0$ and $\text{crit} (\mathcal{M}_4^{01}, \mathcal{M}_3^{01}) = \aleph_2$ (M. Ploščica 2000, 2003) (later extended to unbounded lattices by P. Gillibert);

- $\text{crit} (\mathcal{A}, \mathcal{B}) = \aleph_1$, where $\mathcal{A}$ is generated by the top lattice and $\mathcal{B}$ is generated by the three bottom lattices in the picture below (P. Gillibert 2007).
More specifically,

- $\text{crit}(\mathcal{M}^{01}_3, \mathcal{D}^{01}) = \aleph_0$ and $\text{crit}(\mathcal{M}^{01}_4, \mathcal{M}^{01}_3) = \aleph_2$ (M. Ploščica 2000, 2003) (later extended to unbounded lattices by P. Gillibert);
- $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_1$, where $\mathcal{A}$ is generated by the top lattice and $\mathcal{B}$ is generated by the three bottom lattices in the picture below (P. Gillibert 2007).
Can one go further?

**Question:**

Are there finitely generated lattice varieties $\mathcal{A}$ and $\mathcal{B}$ such that $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_3$?
Can one go further?

**Question:**
Are there finitely generated lattice varieties $\mathcal{A}$ and $\mathcal{B}$ such that $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_3$?

**Answer:** nobody knows so far, but
Can one go further?

**Question:** Are there finitely generated lattice varieties $\mathcal{A}$ and $\mathcal{B}$ such that $\text{crit}(\mathcal{A}, \mathcal{B}) = \aleph_3$?

**Answer:** nobody knows so far, but there’s a feeling that 3-ladders of cardinality $\aleph_2$ could help.
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?
Possible existence of large 3-ladders?

<table>
<thead>
<tr>
<th>Question (S. Z. Ditor 1984)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Does there exist a 3-ladder of cardinality $\aleph_2$?</td>
</tr>
<tr>
<td>Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.</td>
</tr>
</tbody>
</table>
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

Problem: C should be not only a 2-ladder, cofinal in $F_\alpha$
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality \( \aleph_2 \)?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality \( \aleph_2 \).

**Problem:** C should be not only a 2-ladder, cofinal in \( F_\alpha \) (now \( \alpha < \omega_2 \) and \( |F_\alpha| \leq \aleph_1 \)).
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

**Problem:** $C$ should be not only a 2-ladder, cofinal in $F_\alpha$ (now $\alpha < \omega_2$ and $|F_\alpha| \leq \aleph_1$), but also
Possible existence of large 3-ladders?

**Question (S. Z. Ditor 1984)**

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

**Problem:** $C$ should be not only a 2-ladder, cofinal in $F_\alpha$ (now $\alpha < \omega_2$ and $|F_\alpha| \leq \aleph_1$), but also a meet-subsemilattice of $F_\alpha$. 
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

**Problem:** $C$ should be not only a 2-ladder, cofinal in $F_\alpha$ (now $\alpha < \omega_2$ and $|F_\alpha| \leq \aleph_1$), but also a meet-subsemilattice of $F_\alpha$. This is just in order to ensure that $F_{\alpha+1}$ is a lattice.
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

**Problem:** $C$ should be not only a 2-ladder, cofinal in $F_\alpha$ (now $\alpha < \omega_2$ and $|F_\alpha| \leq \aleph_1$), but also a meet-subsemilattice of $F_\alpha$. This is just in order to ensure that $F_{\alpha+1}$ is a lattice.

**Question:**

Let $K$ be a lower finite lattice of cardinality $\leq \aleph_1$. 
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

**Problem:** $C$ should be not only a 2-ladder, cofinal in $F_\alpha$ (now $\alpha < \omega_2$ and $|F_\alpha| \leq \aleph_1$), but also a meet-subsemilattice of $F_\alpha$. This is just in order to ensure that $F_{\alpha+1}$ is a lattice.

**Question:**

Let $K$ be a lower finite lattice of cardinality $\leq \aleph_1$. Does $K$ have a cofinal 2-ladder that is also a meet-subsemilattice of $K$?
Possible existence of large 3-ladders?

Question (S. Z. Ditor 1984)

Does there exist a 3-ladder of cardinality $\aleph_2$?

Try to extend the argument used above for 2-ladders, to the construction of 3-ladders of cardinality $\aleph_2$.

**Problem:** $C$ should be **not only** a 2-ladder, cofinal in $F_\alpha$ (now $\alpha < \omega_2$ and $|F_\alpha| \leq \aleph_1$), **but also** a meet-subsemilattice of $F_\alpha$. This is just in order to ensure that $F_{\alpha+1}$ is a lattice.

**Question:**

Let $K$ be a lower finite lattice of cardinality $\leq \aleph_1$. Does $K$ have a cofinal 2-ladder that is also a meet-subsemilattice of $K$?

**Partial answer (F. Wehrung 2008):**

Yes, provided MA($\aleph_1$; precaliber $\aleph_1$) holds.
A first consistency result for 3-ladders of cardinality $\aleph_2$

Corollary (F. Wehrung 2008):

If $\text{MA}(\aleph_1; \text{precaliber } \aleph_1)$ holds, then there exists a 3-ladder of cardinality $\aleph_2$. 
What is $\text{MA}(\aleph_1; \text{precaliber } \aleph_1)$?

- A subset $X$ in a poset $P$ is centered, if every finite subset of $X$ has a lower bound in $P$ (not necessarily in $X$!).
What is MA(ℵ₁; precaliber ℵ₁)?

- A subset X in a poset P is centered, if every finite subset of X has a lower bound in P (not necessarily in X!).
- A poset P has precaliber ℵ₁, if every uncountable subset of P has an uncountable centered subset.
What is MA(\(\aleph_1\); precaliber \(\aleph_1\))?

- A subset \(X\) in a poset \(P\) is centered, if every finite subset of \(X\) has a lower bound in \(P\) (not necessarily in \(X\)!).
- A poset \(P\) has precaliber \(\aleph_1\), if every uncountable subset of \(P\) has an uncountable centered subset.
- For a collection \(D\) of subsets of \(P\), a filter \(G\) of \(P\) is \(D\)-generic, if \(G \cap D \neq \emptyset\) for each coinitial \(D \in D\).
What is MA(\(\aleph_1;\) precaliber \(\aleph_1\))?

- A subset \(X\) in a poset \(P\) is **centered**, if every finite subset of \(X\) has a lower bound in \(P\) (not necessarily in \(X\!)\).
- A poset \(P\) has **precaliber \(\aleph_1\)**, if every uncountable subset of \(P\) has an uncountable centered subset.
- For a collection \(\mathcal{D}\) of subsets of \(P\), a **filter** \(G\) of \(P\) is **\(\mathcal{D}\)-generic**, if \(G \cap D \neq \emptyset\) for each coinitial \(D \in \mathcal{D}\).
- MA(\(\aleph_1;\) precaliber \(\aleph_1\)) holds, if for every poset \(P\) of precaliber \(\aleph_1\) and every collection \(\mathcal{D}\) of subsets of \(P\), if \(|\mathcal{D}| \leq \aleph_1\), then there exists a \(\mathcal{D}\)-generic filter of \(P\).
What is $\text{MA}(\aleph_1; \text{precaliber } \aleph_1)$?

- A subset $X$ in a poset $P$ is **centered**, if every finite subset of $X$ has a lower bound in $P$ (not necessarily in $X$!).
- A poset $P$ has **precaliber $\aleph_1$**, if every uncountable subset of $P$ has an uncountable centered subset.
- For a collection $\mathcal{D}$ of subsets of $P$, a **filter $G$ of $P$ is $\mathcal{D}$-generic**, if $G \cap D \neq \emptyset$ for each coinitial $D \in \mathcal{D}$.
- $\text{MA}(\aleph_1; \text{precaliber } \aleph_1)$ holds, if for every poset $P$ of precaliber $\aleph_1$ and every collection $\mathcal{D}$ of subsets of $P$, if $|\mathcal{D}| \leq \aleph_1$, then there exists a $\mathcal{D}$-generic filter of $P$.

What about this axiom?

- $\text{MA}(\aleph_1; \text{precaliber } \aleph_1)$ is consistent with ZFC (Solovay and Tennenbaum, 1971).
What is MA($\aleph_1$; precaliber $\aleph_1$)?

- A subset $X$ in a poset $P$ is centered, if every finite subset of $X$ has a lower bound in $P$ (not necessarily in $X$!).
- A poset $P$ has precaliber $\aleph_1$, if every uncountable subset of $P$ has an uncountable centered subset.
- For a collection $\mathcal{D}$ of subsets of $P$, a filter $G$ of $P$ is $\mathcal{D}$-generic, if $G \cap D \neq \emptyset$ for each coinitial $D \in \mathcal{D}$.
- MA($\aleph_1$; precaliber $\aleph_1$) holds, if for every poset $P$ of precaliber $\aleph_1$ and every collection $\mathcal{D}$ of subsets of $P$, if $|\mathcal{D}| \leq \aleph_1$, then there exists a $\mathcal{D}$-generic filter of $P$.

What about this axiom?

- MA($\aleph_1$; precaliber $\aleph_1$) is consistent with ZFC (Solovay and Tennenbaum, 1971).
- MA($\aleph_1$; precaliber $\aleph_1$) implies that $2^{\aleph_0} = 2^{\aleph_1}$ (Martin and Solovay, 1970).
What is \( \text{MA}(\aleph_1; \text{precaliber } \aleph_1) \)?

- A subset \( X \) in a poset \( P \) is **centered**, if every finite subset of \( X \) has a lower bound in \( P \) (not necessarily in \( X \! \)).
- A poset \( P \) has **precaliber \( \aleph_1 \)**, if every uncountable subset of \( P \) has an uncountable centered subset.
- For a collection \( \mathcal{D} \) of subsets of \( P \), a **filter** \( G \) of \( P \) is \( \mathcal{D} \)-**generic**, if \( G \cap D \neq \emptyset \) for each coinitial \( D \in \mathcal{D} \).
- \( \text{MA}(\aleph_1; \text{precaliber } \aleph_1) \) holds, if for every poset \( P \) of precaliber \( \aleph_1 \) and every collection \( \mathcal{D} \) of subsets of \( P \), if \( |\mathcal{D}| \leq \aleph_1 \), then there exists a \( \mathcal{D} \)-generic filter of \( P \).

**What about this axiom?**

- \( \text{MA}(\aleph_1; \text{precaliber } \aleph_1) \) is consistent with ZFC (Solovay and Tennenbaum, 1971).
- \( \text{MA}(\aleph_1; \text{precaliber } \aleph_1) \) implies that \( 2^{\aleph_0} = 2^{\aleph_1} \) (Martin and Solovay, 1970). In particular, it contradicts the Continuum Hypothesis.
Simplified gap-1 morasses

- $\alpha + \beta := \text{sum of two ordinals } \alpha \text{ and } \beta \text{ (non-commutative).}$
Simplified gap-1 morasses

- \( \alpha + \beta \) := sum of two ordinals \( \alpha \) and \( \beta \) (non-commutative).
- \( \beta - \alpha \) := unique ordinal \( \xi \) such that \( \alpha + \xi = \beta \).
Simplified gap-1 morasses

- $\alpha + \beta := \text{sum of two ordinals } \alpha \text{ and } \beta \text{ (non-commutative)}.$
- $\beta - \alpha := \text{unique ordinal } \xi \text{ such that } \alpha + \xi = \beta.$
- For $\alpha \leq \beta,$ define $\tau_{\alpha,\beta}: \beta \to \beta + (\beta - \alpha)$ by

$$
\tau_{\alpha,\beta}(\xi) := \begin{cases} 
\xi, & \text{if } \xi < \alpha, \\
\beta + (\xi - \alpha), & \text{if } \xi \geq \alpha.
\end{cases}
$$
Defining \((\kappa, 1)\)-morasses

**Definition (D. J. Velleman 1984)**

Let \(\kappa\) be an infinite cardinal. A **simplified \((\kappa, 1)\)-morass** is a structure
Defining \((\kappa, 1)\)-morasses

Definition (D. J. Velleman 1984)

Let \(\kappa\) be an infinite cardinal. A simplified \((\kappa, 1)\)-morass is a structure

\[
\mathcal{M} = ((\theta_\alpha \mid \alpha \leq \kappa), (\mathcal{F}_{\alpha, \beta} \mid \alpha < \beta \leq \kappa))
\]
Defining \((\kappa, 1)\)-morasses

**Definition (D. J. Velleman 1984)**

Let \(\kappa\) be an infinite cardinal. A **simplified \((\kappa, 1)\)-morass** is a structure

\[
\mathcal{M} = ((\theta_\alpha \mid \alpha \leq \kappa), (\mathcal{F}_{\alpha,\beta} \mid \alpha < \beta \leq \kappa))
\]

satisfying the following conditions:

(to be continued)
Defining \((k, 1)\)-morasses

**Definition (D. J. Velleman 1984)**

Let \(\kappa\) be an infinite cardinal. A simplified \((k, 1)\)-morass is a structure

\[
\mathcal{M} = \left( (\theta_\alpha \mid \alpha \leq \kappa), (\mathcal{F}_{\alpha,\beta} \mid \alpha < \beta \leq \kappa) \right)
\]

satisfying the following conditions:

(P0) \[(a) \quad \theta_0 = 2, 0 < \theta_\alpha < \kappa \text{ for each } \alpha < \kappa, \text{ and } \theta_\kappa = \kappa^+.
(b) \quad \mathcal{F}_{\alpha,\beta} \text{ is a set of order-embeddings from } \theta_\alpha \text{ into } \theta_\beta, \text{ for all } \alpha < \beta \leq \kappa.
\]

(P1) \[|\mathcal{F}_{\alpha,\beta}| < \kappa, \text{ for all } \alpha < \beta < \kappa.
\]

(P2) \[\text{If } \alpha < \beta < \gamma \leq \kappa, \text{ then } \mathcal{F}_{\alpha,\gamma} = \{ f \circ g \mid f \in \mathcal{F}_{\beta,\gamma} \text{ and } g \in \mathcal{F}_{\alpha,\beta} \}.
\]

(to be continued)
Defining simplified \((κ, 1)\)-morasses (cont’d)

(\text{end of definition of a simplified } (κ, 1)\text{-morass})

(P3) For each \(α < κ\), there exists a \textit{nonzero} ordinal \(δ_α < θ_α\) such that \(θ_{α+1} = θ_α + (θ_α - δ_α)\) and \(F_{α, α+1} = \{id_{θ_α}, τ_{δ_α}, θ_α\}\).

(P4) For every limit ordinal \(λ ≤ κ\), all \(α_i < λ\) and \(f_i ∈ F_{α_i, λ}\), for \(i < 2\), there exists \(α < λ\) with \(α_0, α_1 < α\) together with \(f'_i ∈ F_{α_i, α}\), for \(i < 2\), and \(g ∈ F_{α, λ}\) such that \(f_i = g \circ f'_i\) for each \(i < 2\).

(P5) The equality \(θ_α = \bigcup(f[θ_ξ] \mid ξ < α \text{ and } f ∈ F_{ξ, α})\) holds for each \(α > 0\).
Do these things exist at all?

Theorem

- Simplified \((\omega_1, 1)\)-morasses exist in \(L[A]\), for each \(A \subseteq \omega_1\) (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984).
Do these things exist at all?

Theorem

- Simplified \((\omega_1, 1)\)-morasses exist in \(L[A]\), for each \(A \subseteq \omega_1\) (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984).
- If there exists no simplified \((\omega_1, 1)\)-morass, then \(\omega_2\) is inaccessible in the constructible universe \(L\) (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984).
Do these things exist at all?

Theorem

- Simplified $(\omega_1, 1)$-morasses exist in $L[A]$, for each $A \subseteq \omega_1$ (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984).
- If there exists no simplified $(\omega_1, 1)$-morass, then $\omega_2$ is inaccessible in the constructible universe $L$ (R. Jensen 1970, K. Devlin 1984, and D. J. Velleman 1984).
- If there exists an inaccessible cardinal, then there exists a generic extension without a “Kurepa tree”, and thus without a simplified $(\omega_1, 1)$-morass (J. Silver 1971).
What does this have to do with 3-ladders?

Theorem (F. Wehrung 2008)

If there exists a simplified $(\omega_1, 1)$-morass, then there exists a 3-ladder of cardinality $\aleph_2$.
What does this have to do with 3-ladders?

Theorem (F. Wehrung 2008)
If there exists a simplified \((\omega_1, 1)\)-morass, then there exists a 3-ladder of cardinality \(\aleph_2\).

Corollary
If there exists no 3-ladder of cardinality \(\aleph_2\), then \(\omega_2\) is inaccessible in \(L\).
What does this have to do with 3-ladders?

**Theorem (F. Wehrung 2008)**

If there exists a simplified \((\omega_1, 1)\)-morass, then there exists a 3-ladder of cardinality \(\aleph_2\).

**Corollary**

If there exists no 3-ladder of cardinality \(\aleph_2\), then \(\omega_2\) is inaccessible in \(L\).

**Corollary**

The existence of a 3-ladder of cardinality \(\aleph_2\) is consistent with both the Continuum Hypothesis and its negation.
The question remains:

**Question**

Is the existence of a 3-ladder of cardinality $\aleph_2$ a theorem of ZFC?
The question remains:

**Question**

Is the existence of a 3-ladder of cardinality $\aleph_2$ a theorem of ZFC?

**Eerie situation:** The existence of a 3-ladder of cardinality $\aleph_2$ follows from either one of two axioms that are usually thought of as ‘orthogonal’ to each other.