Approximating the finite by the infinite: Larders and CLL

Friedrich Wehrung

Université de Caen
LMNO, UMR 6139
Département de Mathématiques
14032 Caen cedex
E-mail: wehrung@math.unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung

Most of the results discussed here obtained with Pierre Gillibert.

February 6, 2010
Certain posets $\rightarrow$ lattices

A partially ordered set ($\simeq$poset) $(L, \leq)$ is a lattice, if
A partially ordered set (\(\equiv\) **poset**) \((L, \leq)\) is a **lattice**, if

\[
x \lor y := \sup\{x, y\},
\]

\[
x \land y := \inf\{x, y\}
\]
A partially ordered set (\(\subseteq\text{poset}\)) \((L, \leq)\) is a **lattice**, if

\[
x \lor y := \sup\{x, y\},
\]
\[
x \land y := \inf\{x, y\}
\]

exist for all \(x, y \in L\).
Certain posets $\to$ lattices

A partially ordered set (\textit{\textasciitilde poset}) $(L, \leq)$ is a \textit{lattice}, if

$$x \lor y := \sup\{x, y\},$$
$$x \land y := \inf\{x, y\}$$

exist for all $x, y \in L$. The following are valid in all lattices:
A partially ordered set (\( \equiv \text{poset} \) \((L, \leq)\)) is a lattice, if

\[
x \lor y := \sup\{x, y\},
\]
\[
x \land y := \inf\{x, y\}
\]

exist for all \( x, y \in L \). The following are valid in all lattices:

\[
(x \lor y) \lor z = x \lor (y \lor z); \quad x \lor y = y \lor x; \quad x \lor x = x;
\]
\[
(x \land y) \land z = x \land (y \land z); \quad x \land y = y \land x; \quad x \land x = x
\]
Certain posets $\to$ lattices

A partially ordered set (\(\text{poset}\)) \((L, \leq)\) is a lattice, if

\[
x \lor y := \text{sup}\{x, y\},
\]
\[
x \land y := \text{inf}\{x, y\}
\]

exist for all \(x, y \in L\). The following are valid in all lattices:

\[
(x \lor y) \lor z = x \lor (y \lor z); \quad x \lor y = y \lor x; \quad x \lor x = x;
\]
\[
(x \land y) \land z = x \land (y \land z); \quad x \land y = y \land x; \quad x \land x = x
\]

(\text{semilattice laws}), and
Certain posets $\rightarrow$ lattices

A partially ordered set (=$\text{poset}$) $(L, \leq)$ is a lattice, if

$$x \lor y := \sup\{x, y\},$$
$$x \land y := \inf\{x, y\}$$

exist for all $x, y \in L$. The following are valid in all lattices:

$$(x \lor y) \lor z = x \lor (y \lor z); \quad x \lor y = y \lor x; \quad x \lor x = x;$$
$$(x \land y) \land z = x \land (y \land z); \quad x \land y = y \land x; \quad x \land x = x$$

(semilattice laws), and

$$x \lor (x \land y) = x \land (x \lor y) = x$$
A partially ordered set (\(\subseteq \text{poset}\)) \((L, \leq)\) is a lattice, if

\[
x \lor y := \sup\{x, y\}, \quad x \land y := \inf\{x, y\}
\]

exist for all \(x, y \in L\). The following are valid in all lattices:

\[
\begin{align*}
(x \lor y) \lor z &= x \lor (y \lor z); \\
\quad x \lor y &= y \lor x; \\
\quad x \lor x &= x;
\end{align*}
\]

\[
\begin{align*}
(x \land y) \land z &= x \land (y \land z); \\
\quad x \land y &= y \land x; \\
\quad x \land x &= x
\end{align*}
\]

(\text{semilattice laws}), and

\[
x \lor (x \land y) = x \land (x \lor y) = x
\]

(\text{absorption laws}).
Certain posets $\rightarrow$ lattices

A partially ordered set (\textit{\textbf{poset}}) $(L, \leq)$ is a \textit{lattice}, if

$$x \lor y := \sup \{x, y\},$$
$$x \land y := \inf \{x, y\}$$

exist for all $x, y \in L$. The following are valid in all lattices:

$$(x \lor y) \lor z = x \lor (y \lor z); \quad x \lor y = y \lor x; \quad x \lor x = x;$$
$$(x \land y) \land z = x \land (y \land z); \quad x \land y = y \land x; \quad x \land x = x$$

(\textit{semilattice laws}), and

$$x \lor (x \land y) = x \land (x \lor y) = x$$

(\textit{absorption laws}). We also say that $(L, \lor, \land)$ is a \textit{lattice}. 
Conversely, if \((L, \vee, \wedge)\) satisfies the axioms (semilattice, absorption) above, define a binary relation \(\leq\) on \(L\) by

\[
x \leq y \iff x \vee y = y, \quad x \wedge y = x.
\]

Then \(\leq\) is a partial ordering, and \(x \vee y = \text{sup} \{x, y\}\), \(x \wedge y = \text{inf} \{x, y\}\) with respect to that partial ordering.
Conversely, if \((L, \lor, \land)\) satisfies the axioms (semilattice, absorption) above, define a binary relation \(\leq\) on \(L\) by

\[
x \leq y \iff x \lor y = y, \quad \iff x \land y = x.
\]
Lattices $\rightarrow$ certain posets

Conversely, if $(L, \vee, \wedge)$ satisfies the axioms (semilattice, absorption) above, define a binary relation $\leq$ on $L$ by

$$x \leq y \iff x \vee y = y,$$

$$\iff x \wedge y = x.$$

Then $\leq$ is a partial ordering, and
Conversely, if \((L, \vee, \wedge)\) satisfies the axioms (semilattice, absorption) above, define a binary relation \(\leq\) on \(L\) by

\[
x \leq y \iff x \vee y = y, \\
\iff x \wedge y = x.
\]

Then \(\leq\) is a partial ordering, and \(x \vee y = \text{sup}\{x, y\}\), \(x \wedge y = \text{inf}\{x, y\}\) with respect to that partial ordering.
Lattices $\rightarrow$ certain posets

Conversely, if $(L, \lor, \land)$ satisfies the axioms (semilattice, absorption) above, define a binary relation $\leq$ on $L$ by

$$x \leq y \iff x \lor y = y,$$
$$\iff x \land y = x.$$ 

Then $\leq$ is a partial ordering, and $x \lor y = \sup\{x, y\}$, $x \land y = \inf\{x, y\}$ with respect to that partial ordering.

Hasse diagrams of the lattices $M_3$ and $N_5$: 

Latters and
CLL

Lattices, congruences, varieties

Critical points between varieties

General settings; CLL

Coordinatization of lattices by regular rings

Non-coordinatizable SCMLs

Lattices without CPCP-extension
Lattices → certain posets

Conversely, if \((L, \lor, \land)\) satisfies the axioms (semilattice, absorption) above, define a binary relation \(\leq\) on \(L\) by

\[
x \leq y \iff x \lor y = y, \quad x \land y = x.
\]

Then \(\leq\) is a partial ordering, and \(x \lor y = \sup\{x, y\}\), \(x \land y = \inf\{x, y\}\) with respect to that partial ordering.

Hasse diagrams of the lattices \(M_3\) and \(N_5\):

\[\text{Diagram of } M_3\]
\[\text{Diagram of } N_5\]
A lattice is **distributive** if it satisfies the identity

\[
x \land (y \lor z) = (x \land y) \lor (x \land z).
\]

This identity is self-dual (not affected by \(\lor \leftrightarrow \land\)).

A lattice is **modular** if it satisfies the quasi-identity

\[
x \geq z \Rightarrow x \land (y \lor z) = (x \land y) \lor z.
\]

This is equivalent to the identity

\[
x \land (y \lor (x \land z)) = (x \land y) \lor (x \land z).
\]

Modularity is also self-dual. It is implied by distributivity.

A lattice is modular (resp., distributive) iff it contains no copy of \(N_5\) (resp., \(M_3\) and \(N_5\)).
A lattice is **distributive** if it satisfies the identity
\[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]
Distributive, modular...

- A lattice is **distributive** if it satisfies the identity
  \[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

- This identity is **self-dual** (not affected by \( \lor \Leftrightarrow \land \)).
A lattice is **distributive** if it satisfies the identity
\[ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \]

This identity is **self-dual** (not affected by \( \vee \leftrightarrow \wedge \)).

A lattice is **modular** if it satisfies the **quasi-identity**
Distributive, modular...

- A lattice is **distributive** if it satisfies the identity
  \[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

- This identity is **self-dual** (not affected by \( \lor \iff \land \)).
- A lattice is **modular** if it satisfies the quasi-identity
  \[ x \geq z \Rightarrow x \land (y \lor z) = (x \land y) \lor z. \]
A lattice is distributive if it satisfies the identity
\[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

This identity is self-dual (not affected by \( \lor \leftrightarrow \land \)).

A lattice is modular if it satisfies the quasi-identity
\[ x \geq z \Rightarrow x \land (y \lor z) = (x \land y) \lor z. \]

This is equivalent to the identity
A lattice is **distributive** if it satisfies the identity

\[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

This identity is **self-dual** (not affected by \( \lor \Leftrightarrow \land \)).

A lattice is **modular** if it satisfies the quasi-identity

\[ x \geq z \implies x \land (y \lor z) = (x \land y) \lor z. \]

This is equivalent to the identity

\[ x \land (y \lor (x \land z)) = (x \land y) \lor (x \land z). \]
Distributive, modular...

- A lattice is **distributive** if it satisfies the identity
  \[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

- This identity is **self-dual** (not affected by \( \lor \equiv \land \)).

- A lattice is **modular** if it satisfies the **quasi-identity**
  \[ x \geq z \implies x \land (y \lor z) = (x \land y) \lor z. \]

- This is equivalent to the **identity**
  \[ x \land (y \lor (x \land z)) = (x \land y) \lor (x \land z). \]

- Modularity is also self-dual. It is implied by distributivity.
A lattice is **distributive** if it satisfies the identity

\[ x \land (y \lor z) = (x \land y) \lor (x \land z). \]

This identity is **self-dual** (not affected by \( \lor \leftrightarrow \land \)).

A lattice is **modular** if it satisfies the quasi-identity

\[ x \geq z \implies x \land (y \lor z) = (x \land y) \lor z. \]

This is equivalent to the **identity**

\[ x \land (y \lor (x \land z)) = (x \land y) \lor (x \land z). \]

Modularity is also self-dual. It is implied by distributivity.

A lattice is modular (resp., distributive) iff it contains no copy of \( N_5 \) (resp., \( M_3 \) and \( N_5 \)).
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. 

---

**Larders and CLL**

- Lattices, congruences, varieties
- Critical points between varieties
- General settings; CLL
- Coordinatization of lattices by regular rings
- Non-coordinatizable SCMLs
- Lattices without CPCP-extension
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \vee y = x \cup y$, $x \wedge y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).

- $\mathcal{C}(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x) \\forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. **Distributive**.
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).

- $C(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x)$ $\forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. **Distributive**.

- For a **group** $G$,
Examples of lattices

- The powerset \( \mathcal{P}(X) \) of a set \( X \), with \( \subseteq \). There, \( x \lor y = x \cup y \), \( x \land y = x \cap y \); **distributive**. Every distributive lattice is contained in some \( \mathcal{P}(X) \) (Birkhoff, Stone).
- \( C(X, \mathbb{R}) \), \( X \) a topological space, with \( f \leq g \) iff \( f(x) \leq g(x) \ \forall x \in X \). Then \( (f \lor g)(x) = \max\{f(x), g(x)\} \), \( (f \land g)(x) = \min\{f(x), g(x)\} \). **Distributive**.
- For a group \( G \),
  \[ \text{NSub } G := \{X \mid X \text{ is a normal subgroup of } G\} \text{.} \]
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).

- $\mathcal{C}(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x) \ \forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. **Distributive**.

- For a **group** $G$,

  $$\text{NSub } G := \{X \mid X \text{ is a normal subgroup of } G\}.$$  

  **Modular**.
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).

- $C(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x)$ $\forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. **Distributive**.

- For a group $G$, $\text{NSub } G := \{X \mid X \text{ is a normal subgroup of } G\}$. **Modular**. If “normal” removed, then no identity.
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).
- $\mathcal{C}(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x)$ $\forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. **Distributive**.
- For a group $G$,
  \[
  \text{NSub } G := \{X \mid X \text{ is a normal subgroup of } G\}.
  \]
  **Modular**. If “normal” removed, then no identity.
- For a module $M$ over a ring $R$,
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; distributive. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).

- $\mathcal{C}(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x) \ \forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. Distributive.

- For a group $G$,
  \[ \text{NSub } G := \{X \mid X \text{ is a normal subgroup of } G\} \]
  Modular. If “normal” removed, then no identity.

- For a module $M$ over a ring $R$,
  \[ \text{Sub } M := \{X \mid X \text{ is a submodule of } M\} \]
Examples of lattices

- The powerset $\mathcal{P}(X)$ of a set $X$, with $\subseteq$. There, $x \lor y = x \cup y$, $x \land y = x \cap y$; **distributive**. Every distributive lattice is contained in some $\mathcal{P}(X)$ (Birkhoff, Stone).

- $\mathcal{C}(X, \mathbb{R})$, $X$ a topological space, with $f \leq g$ iff $f(x) \leq g(x) \ \forall x \in X$. Then $(f \lor g)(x) = \max\{f(x), g(x)\}$, $(f \land g)(x) = \min\{f(x), g(x)\}$. **Distributive**.

- For a group $G$,

  $$\text{NSub } G := \{X \mid X \text{ is a normal subgroup of } G\}.$$  

  **Modular.** If “normal” removed, then no identity.

- For a module $M$ over a ring $R$,

  $$\text{Sub } M := \{X \mid X \text{ is a submodule of } M\}.$$  

  **Modular.** Particular case: subspace lattices of **vector spaces**.
Further examples of lattices

- The lattice $\text{Eq } X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. Not modular, no identity ($X$ infinite).
Further examples of lattices

- The lattice $\text{Eq } X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. Not modular, no identity ($X$ infinite).
- For permutations $\alpha$ and $\beta$ on $\{1, \ldots, n\}$, set
  \[
  \text{Inv}(\alpha) := \{ (i, j) \mid i < j \text{ and } \alpha(i) > \alpha(j) \},
  \]
  
  \[
  \alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta).
  \]
Further examples of lattices

- The lattice $\text{Eq} \ X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. Not modular, no identity ($X$ infinite).
- For permutations $\alpha$ and $\beta$ on $\{1, \ldots, n\}$, set
  \[
  \text{Inv}(\alpha) := \{(i, j) \mid i < j \text{ and } \alpha(i) > \alpha(j)\},
  \]
  \[
  \alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta).
  \]
  We get the permutohedron on $n$ letters. Not modular for $n \geq 3$. Any identity for all of them? Open problem.
Further examples of lattices

- The lattice $\text{Eq } X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. Not modular, no identity ($X$ infinite).
- For permutations $\alpha$ and $\beta$ on $\{1, \ldots, n\}$, set
  \[
  \text{Inv}(\alpha) := \{(i, j) \mid i < j \text{ and } \alpha(i) > \alpha(j)\},
  \]
  \[\alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta).\]

  We get the permutohedron on $n$ letters. Not modular for $n \geq 3$. Any identity for all of them? Open problem.
- A subset $X$ in a poset $P$ is order-convex if $x \leq y \leq z$ and $x, z \in X$ implies that $y \in X$. 
Further examples of lattices

- The lattice $\text{Eq } X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. **Not modular, no identity** ($X$ infinite).

- For permutations $\alpha$ and $\beta$ on $\{1, \ldots, n\}$, set
  \[ \text{Inv}(\alpha) := \{(i, j) \mid i < j \text{ and } \alpha(i) > \alpha(j)\}, \]
  \[ \alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta). \]

  We get the **permutohedron** on $n$ letters. **Not modular** for $n \geq 3$. Any identity for all of them? **Open problem.**

- A subset $X$ in a poset $P$ is **order-convex** if $x \leq y \leq z$ and $x, z \in X$ implies that $y \in X$.

  \[ \text{Co}(P) := \{X \subseteq P \mid X \text{ is order-convex}\}, \quad \text{with } \subseteq. \]
Further examples of lattices

- The lattice $\text{Eq} \ X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. Not modular, no identity ($X$ infinite).
- For permutations $\alpha$ and $\beta$ on $\{1, \ldots, n\}$, set
  \[
  \text{Inv}(\alpha) := \{(i, j) \mid i < j \text{ and } \alpha(i) > \alpha(j)\},
  \]
  \[
  \alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta).
  \]
  We get the permutohedron on $n$ letters. Not modular for $n \geq 3$. Any identity for all of them? Open problem.
- A subset $X$ in a poset $P$ is order-convex if $x \leq y \leq z$ and $x, z \in X$ implies that $y \in X$.
  \[
  \text{Co}(P) := \{X \subseteq P \mid X \text{ is order-convex}\}, \quad \text{with} \quad \subseteq.
  \]
  Not modular as a rule, but has other identities, such as
Further examples of lattices

- The lattice $\text{Eq } X$ of all equivalence relations on a set $X$, ordered by $\subseteq$. Not modular, no identity ($X$ infinite).
- For permutations $\alpha$ and $\beta$ on $\{1, \ldots, n\}$, set
  \[ \text{Inv}(\alpha) := \{(i, j) \mid i < j \text{ and } \alpha(i) > \alpha(j)\}, \]
  \[ \alpha \leq \beta \iff \text{Inv}(\alpha) \subseteq \text{Inv}(\beta). \]
  
  We get the permutohedron on $n$ letters. Not modular for $n \geq 3$. Any identity for all of them? Open problem.
- A subset $X$ in a poset $P$ is order-convex if $x \leq y \leq z$ and $x, z \in X$ implies that $y \in X$.
  \[ \text{Co}(P) := \{X \subseteq P \mid X \text{ is order-convex}\}, \text{ with } \subseteq . \]
  
  Not modular as a rule, but has other identities, such as
  \[ x \land (x_0 \lor x_1) \land (x_1 \lor x_2) \land (x_0 \lor x_2) \]
  \[ = (x \land x_0 \land (x_1 \lor x_2)) \lor (x \land x_1 \land (x_0 \lor x_2)) \lor (x \land x_2 \land (x_0 \lor x_1)). \]
A **variety** is the class of all structures (here, lattices) that satisfy a given set of identities.
A **variety** is the class of all structures (here, lattices) that satisfy a given set of identities. For example, $\mathcal{L}$ is the variety of all lattices, $\mathcal{M}$ is the variety of all modular lattices, $\mathcal{N}_5$ is the variety generated by $\mathcal{N}_5$, ...
A variety is the class of all structures (here, lattices) that satisfy a given set of identities. For example, $\mathcal{L}$ is the variety of all lattices, $\mathcal{M}$ is the variety of all modular lattices, $\mathcal{N}_5$ is the variety generated by $\mathcal{N}_5$, . . . Finitely generated variety of lattices: generated by a finite lattice.
Variety is the spice of life

A variety is the class of all structures (here, lattices) that satisfy a given set of identities. For example, $\mathcal{L}$ is the variety of all lattices, $\mathcal{M}$ is the variety of all modular lattices, $\mathcal{N}_5$ is the variety generated by $\mathcal{N}_5$, . . . Finitely generated variety of lattices: generated by a finite lattice. (Very) partial picture of the lattice of all varieties of lattices:
Congruences, congruence lattices

- **Congruence** of a lattice $L$: equivalence relation $\theta$ on $L$, compatible with both $\lor$ and $\land$ operations:
Congruences, congruence lattices

- **Congruence** of a lattice $L$: equivalence relation $\theta$ on $L$, compatible with both $\vee$ and $\wedge$ operations:

  $$x \equiv_\theta y \iff (x \vee z \equiv_\theta y \vee z \text{ and } x \wedge z \equiv_\theta y \wedge z).$$
Congruences, congruence lattices

- **Congruence** of a lattice $L$: equivalence relation $\theta$ on $L$, compatible with both $\lor$ and $\land$ operations:

  $$x \equiv_\theta y \implies (x \lor z \equiv_\theta y \lor z \text{ and } x \land z \equiv_\theta y \land z).$$

  Then set $\text{Con } L := \{\theta \mid \theta \text{ is a congruence of } L\}$. 
**Congruences, congruence lattices**

- **Congruence** of a lattice $L$: equivalence relation $\theta$ on $L$, compatible with both $\lor$ and $\land$ operations:

  $$x \equiv_\theta y \implies (x \lor z \equiv_\theta y \lor z \text{ and } x \land z \equiv_\theta y \land z).$$

  Then set $\text{Con } L := \{\theta \mid \theta \text{ is a congruence of } L\}$.

- Ordered by $\alpha \leq \beta \iff \alpha \subseteq \beta$. 

---

**Ladders and CLL**

- Critical points between varieties
- General settings; CLS
- Coordinatization of lattices by regular rings
- Non-coordinatizable SCMLs
- Lattices without CPCP-extension
Congruences, congruence lattices

- **Congruence** of a lattice $L$: equivalence relation $\theta$ on $L$, compatible with both $\lor$ and $\land$ operations:

  $$x \equiv_{\theta} y \iff (x \lor z \equiv_{\theta} y \lor z \text{ and } x \land z \equiv_{\theta} y \land z).$$

  Then set $\text{Con } L := \{\theta \mid \theta \text{ is a congruence of } L\}$.

- Ordered by $\alpha \leq \beta \iff \alpha \subseteq \beta$. Then $\text{Con } L$, under $\subseteq$, is an “algebraic” lattice (**nothing special about lattices here**). It is also a **distributive** lattice.
Congruences, congruence lattices

- **Congruence** of a lattice \( L \): equivalence relation \( \theta \) on \( L \), compatible with both \( \lor \) and \( \land \) operations:

\[
    x \equiv_{\theta} y \implies (x \lor z \equiv_{\theta} y \lor z \text{ and } x \land z \equiv_{\theta} y \land z).
\]

Then set \( \text{Con}_L := \{ \theta \mid \theta \text{ is a congruence of } L \} \).

- Ordered by \( \alpha \leq \beta \iff \alpha \subseteq \beta \). Then \( \text{Con}_L \), under \( \subseteq \), is an “algebraic” lattice (nothing special about lattices here). It is also a **distributive** lattice. This is very particular to lattices.
Congruences, congruence lattices

- **Congruence** of a lattice $L$: equivalence relation $\theta$ on $L$, compatible with both $\lor$ and $\land$ operations:

  $$x \equiv_\theta y \implies (x \lor z \equiv_\theta y \lor z \text{ and } x \land z \equiv_\theta y \land z).$$

  Then set $\text{Con} \ L := \{\theta \mid \theta \text{ is a congruence of } L\}$.

- **Ordered by** $\alpha \leq \beta \iff \alpha \subseteq \beta$. Then $\text{Con} \ L$, under $\subseteq$, is an “algebraic” lattice (nothing special about lattices here). It is also a **distributive** lattice. This is very particular to lattices.

- **Finitely generated** (=$\text{compact}$) congruence: least congruence that identifies $x_1$ with $y_1$, $\ldots$, $x_n$ with $y_n$ (where $x_i, y_i \in L$ given).
Congruence classes; critical points

- **Congruence class** of a variety $\mathcal{V}$: $\text{Con } \mathcal{V} := \text{class of all lattices isomorphic to some } \text{Con } L$, where $L \in \mathcal{V}$. Fully understood only for $\mathcal{V} = \text{either } \mathcal{I} \text{ or } \mathcal{D}$. 
Congruence classes; critical points

- **Congruence class** of a variety $\mathcal{V}$: $\text{Con} \mathcal{V} := \text{class of all lattices isomorphic to some } \text{Con} L$, where $L \in \mathcal{V}$. **Fully understood only for $\mathcal{V} =$ either $\mathcal{T}$ or $\mathcal{D}$.**

- **Critical point** $\text{crit}(\mathcal{A}; \mathcal{B})$, for varieties $\mathcal{A}$ and $\mathcal{B}$: least possible number of compact elements of a member of $\text{Con} \mathcal{A}$ not in $\text{Con} \mathcal{B}$. 

Valid for varieties of other structures than lattices. Measures the inclusion defect of $\text{Con} \mathcal{A}$ into $\text{Con} \mathcal{B}$. The larger the critical point, the more $\text{Con} \mathcal{A}$ is contained in $\text{Con} \mathcal{B}$.

Example: $\text{crit}(\text{groups}; \text{lattices}) = 5$. On the other hand, $\text{crit}(\text{lattices}; \text{groups}) = \aleph_2$ (Růžička, Tůma, and W.).
Congruence classes; critical points

- **Congruence class** of a variety $\mathcal{V}$: $\text{Con} \mathcal{V} :=$ class of all lattices isomorphic to some $\text{Con} \mathcal{L}$, where $\mathcal{L} \in \mathcal{V}$. **Fully understood only for $\mathcal{V} =$ either $\mathcal{T}$ or $\mathcal{D}$.**

- **Critical point** $\text{crit}(\mathcal{A}; \mathcal{B})$, for varieties $\mathcal{A}$ and $\mathcal{B}$: least possible number of compact elements of a member of $\text{Con} \mathcal{A}$ not in $\text{Con} \mathcal{B}$.

- Valid for varieties of other structures than lattices.
Congruence classes; critical points

- **Congruence class** of a variety $\mathcal{V}$: $\text{Con} \mathcal{V} :=$ class of all lattices isomorphic to some $\text{Con} L$, where $L \in \mathcal{V}$. **Fully understood only for $\mathcal{V} =$ either $\mathcal{I}$ or $\mathcal{D}$.**

- **Critical point** $\text{crit}(\mathcal{A}; \mathcal{B})$, for varieties $\mathcal{A}$ and $\mathcal{B}$: least possible number of compact elements of a member of $\text{Con} \mathcal{A}$ not in $\text{Con} \mathcal{B}$.

- Valid for varieties of other structures than lattices.

- Measures the **inclusion defect** of $\text{Con} \mathcal{A}$ into $\text{Con} \mathcal{B}$. The larger the critical point, the more $\text{Con} \mathcal{A}$ is contained in $\text{Con} \mathcal{B}$.
Congruence classes; critical points

- **Congruence class** of a variety \( \mathcal{V} \): \( \text{Con} \mathcal{V} := \) class of all lattices isomorphic to some \( \text{Con} \mathcal{L} \), where \( \mathcal{L} \in \mathcal{V} \). Fully understood only for \( \mathcal{V} = \) either \( \mathcal{T} \) or \( \mathcal{D} \).

- **Critical point** \( \text{crit} (\mathcal{A}; \mathcal{B}) \), for varieties \( \mathcal{A} \) and \( \mathcal{B} \): least possible number of compact elements of a member of \( \text{Con} \mathcal{A} \) not in \( \text{Con} \mathcal{B} \).

- Valid for varieties of other structures than lattices.

- Measures the **inclusion defect** of \( \text{Con} \mathcal{A} \) into \( \text{Con} \mathcal{B} \). The larger the critical point, the more \( \text{Con} \mathcal{A} \) is contained in \( \text{Con} \mathcal{B} \).

- **Example**: \( \text{crit} (\text{groups}, \text{lattices}) = 5 \). On the other hand, \( \text{crit} (\text{lattices}, \text{groups}) = \aleph_2 \) (Růžička, Tůma, and W.).
Critical points are difficult to calculate

**Notation:** $\text{Var}(L) :=$ variety generated by $L$. It is the class of all lattices satisfying all identities satisfied by $L$. 
Critical points are difficult to calculate

**Notation:** \( \text{Var}(L) := \text{variety generated by } L \). It is the class of all lattices satisfying all identities satisfied by \( L \).

**Theorem (Gillibert 2007)**
Critical points are difficult to calculate

**Notation:** $\text{Var}(L) :=$ variety generated by $L$. It is the class of all lattices satisfying all identities satisfied by $L$.

**Theorem (Gillibert 2007)**

For any finite lattices $A$ and $B$ with $A \notin \text{Var}(B)$, either $\text{crit}(\text{Var}(A); \text{Var}(B))$ is finite or $\text{crit}(\text{Var}(A); \text{Var}(B)) = \aleph_n$ for some $n$. 

Open problem: Let $\gamma(A, B) :=$ least $n$ such that $\text{crit}(\text{Var}(A); \text{Var}(B)) \leq \aleph_n$, for finite lattices $A$ and $B$. Is $\gamma$ recursive?

Examples were known with $n = 0$ and $n = 2$ (M. Ploščica). Later, P. Gillibert found an example with $n = 1$. Recently, P. Gillibert proved that $n \in \{0, 1, 2\}$.
**Critical points are difficult to calculate**

**Notation:** \( \text{Var}(L) := \text{variety generated by } L \). It is the class of all lattices satisfying all identities satisfied by \( L \).

**Theorem (Gillibert 2007)**

For any finite lattices \( A \) and \( B \) with \( A \not\in \text{Var}(B) \), either \( \text{crit}(\text{Var}(A); \text{Var}(B)) \) is finite or \( \text{crit}(\text{Var}(A); \text{Var}(B)) = \aleph_n \) for some \( n \).

**Open problem:**

Let \( \gamma(A, B) := \text{least } n \text{ such that } \text{crit}(\text{Var}(A); \text{Var}(B)) \leq \aleph_n \), for finite lattices \( A \) and \( B \). Is \( \gamma \) recursive? Examples were known with \( n = 0 \) and \( n = 2 \) (M. Ploščica). Later, P. Gillibert found an example with \( n = 1 \). Recently, P. Gillibert proved that \( n \in \{0, 1, 2\} \).
Critical points are difficult to calculate

Notation: \( \text{Var}(L) := \text{variety generated by } L \). It is the class of all lattices satisfying all identities satisfied by \( L \).

Theorem (Gillibert 2007)

For any finite lattices \( A \) and \( B \) with \( A \notin \text{Var}(B) \), either \( \text{crit}(\text{Var}(A); \text{Var}(B)) \) is finite or \( \text{crit}(\text{Var}(A); \text{Var}(B)) = \aleph_n \) for some \( n \).

Open problem:

Let \( \gamma(A, B) := \text{least } n \) such that \( \text{crit}(\text{Var}(A); \text{Var}(B)) \leq \aleph_n \), for finite lattices \( A \) and \( B \). Is \( \gamma \) recursive?
Critical points are difficult to calculate

**Notation:** \( \text{Var}(L) := \text{variety generated by } L \). It is the class of all lattices satisfying all identities satisfied by \( L \).

**Theorem (Gillibert 2007)**

For any finite lattices \( A \) and \( B \) with \( A \not\in \text{Var}(B) \), either \( \text{crit}(\text{Var}(A); \text{Var}(B)) \) is finite or \( \text{crit}(\text{Var}(A); \text{Var}(B)) = \aleph_n \) for some \( n \).

**Open problem:**

Let \( \gamma(A, B) := \text{least } n \) such that \( \text{crit}(\text{Var}(A); \text{Var}(B)) \leq \aleph_n \), for finite lattices \( A \) and \( B \). Is \( \gamma \) recursive?

Examples were known with \( n = 0 \) and \( n = 2 \) (M. Ploščica).
Critical points are difficult to calculate

**Notation:** $\text{Var}(L) :=$ variety generated by $L$. It is the class of all lattices satisfying all identities satisfied by $L$.

**Theorem (Gillibert 2007)**

For any finite lattices $A$ and $B$ with $A \notin \text{Var}(B)$, either $\text{crit}(\text{Var}(A); \text{Var}(B))$ is finite or $\text{crit}(\text{Var}(A); \text{Var}(B)) = \aleph_n$ for some $n$.

**Open problem:**

Let $\gamma(A, B) :=$ least $n$ such that $\text{crit}(\text{Var}(A); \text{Var}(B)) \leq \aleph_n$, for finite lattices $A$ and $B$. Is $\gamma$ recursive?

Examples were known with $n = 0$ and $n = 2$ (M. Ploščica). Later, P. Gillibert found an example with $n = 1$. 


Critical points are difficult to calculate

**Notation:** $\text{Var}(L)$ := variety generated by $L$. It is the class of all lattices satisfying all identities satisfied by $L$.

**Theorem (Gillibert 2007)**
For any finite lattices $A$ and $B$ with $A \notin \text{Var}(B)$, either $\text{crit}([\text{Var}(A); \text{Var}(B)])$ is finite or $\text{crit}([\text{Var}(A); \text{Var}(B)]) = \aleph_n$ for some $n$.

**Open problem:**
Let $\gamma(A, B) := \text{least } n \text{ such that } \text{crit}([\text{Var}(A); \text{Var}(B)]) \leq \aleph_n$, for finite lattices $A$ and $B$. Is $\gamma$ recursive?

Examples were known with $n = 0$ and $n = 2$ (M. Ploščica). Later, P. Gillibert found an example with $n = 1$. Recently, P. Gillibert proved that $n \in \{0, 1, 2\}$. 
We are given finite (or, more generally, algebraic) distributive lattices $S$ and $T$, and a $(\lor, 0)$-homomorphism $\varphi: S \to T$. 
Lifting an arrow between congruence lattices

- We are given finite (or, more generally, algebraic) distributive lattices $S$ and $T$, and a $(\lor, 0)$-homo-morphism $\varphi: S \to T$.
- We want to represent $\varphi: S \to T$ as
  $\text{Con } f : \text{Con } A \to \text{Con } B$, for lattices $A$ and $B$ [in a given variety] and a lattice homomorphism $f : A \to B$. 
Lifting an arrow between congruence lattices

- We are given finite (or, more generally, algebraic) distributive lattices $S$ and $T$, and a $(\vee, 0)$-homomorphism $\varphi: S \to T$.

- We want to represent $\varphi: S \to T$ as
  \[ \text{Con } f : \text{Con } A \to \text{Con } B, \]
  for lattices $A$ and $B$ [in a given variety] and a lattice homomorphism $f: A \to B$.

- **Technical prerequisite:** the assignment $A \mapsto \text{Con } A$ can also be nicely extended to homomorphisms (i.e., defining $\text{Con } f$).
Lifting an arrow between congruence lattices

- We are given finite (or, more generally, algebraic) distributive lattices $S$ and $T$, and a $(\vee, 0)$-homomorphism $\varphi: S \to T$.

- We want to represent $\varphi: S \to T$ as $\text{Con } f: \text{Con } A \to \text{Con } B$, for lattices $A$ and $B$ [in a given variety] and a lattice homomorphism $f: A \to B$.

- Technical prerequisite: the assignment $A \mapsto \text{Con } A$ can also be nicely extended to homomorphisms (i.e., defining $\text{Con } f$). Means that $A \mapsto \text{Con } A$, $f \mapsto \text{Con } f$ is a functor.
Lifting an arrow between congruence lattices

- We are given finite (or, more generally, algebraic) distributive lattices $S$ and $T$, and a $(\lor, 0)$-homo-morphism $\varphi : S \to T$.

- We want to represent $\varphi : S \to T$ as $\text{Con } f : \text{Con } A \to \text{Con } B$, for lattices $A$ and $B$ [in a given variety] and a lattice homomorphism $f : A \to B$.

- Technical prerequisite: the assignment $A \mapsto \text{Con } A$ can also be nicely extended to homomorphisms (i.e., defining $\text{Con } f$). Means that $A \mapsto \text{Con } A$, $f \mapsto \text{Con } f$ is a functor. Straightforward.
Lifting an arrow (continued)

- **Back to the problem with one arrow**: we need lattices $A$ and $B$, a homomorphism $f: A \rightarrow B$, and a “commutative diagram”
Lifting an arrow (continued)

- **Back to the problem with one arrow:** we need lattices $A$ and $B$, a homomorphism $f : A \to B$, and a “commutative diagram”

\[
\begin{array}{ccc}
\text{Con } A & \overset{\text{Con } f}{\longrightarrow} & \text{Con } B \\
\downarrow & & \downarrow \\
S & \overset{\varphi}{\longrightarrow} & T
\end{array}
\]

- We say that $f : A \to B$ lifts $\varphi : S \to T$. 
Lifting an arrow (continued)

- **Back to the problem with one arrow:** we need lattices $A$ and $B$, a homomorphism $f : A \rightarrow B$, and a “commutative diagram”

\[
\begin{array}{ccc}
\text{Con } A & \xrightarrow{\text{Con } f} & \text{Con } B \\
\downarrow \phi & & \downarrow \phi \\
S & \xrightarrow{\varphi} & T
\end{array}
\]

- We say that $f : A \rightarrow B$ lifts $\varphi : S \rightarrow T$.
- Lifting problems: can also be defined for more complex diagrams of finite distributive lattices and $(\lor, 0)$-homomorphisms.
Gillibert’s starting point for the critical point $\aleph_1$

Guess the finite lattices $A$ and $B$:
Gillibert’s starting point for the critical point $\aleph_1$
How Gillibert proceeds for the critical point \( \aleph_1 \)

- **Guess** a finite diagram, of finite distributive lattices and \((\lor, 0)\)-homomorphisms:
How Gillibert proceeds for the critical point $\aleph_1$

- **Guess** a finite diagram, of finite distributive lattices and $(\lor, 0)$-homomorphisms:

```
  {0, 1}^2  \{0, 1\}  \{0, 1\}^2
  \downarrow\psi \uparrow\psi
  \{0, 1\}  \{0, 1\}^2
  \downarrow\varphi_1 \uparrow\varphi_2
  \{0, 1\}^4
```

Prove that the diagram can be lifted in $\text{Var}(A)$, but not in $\text{Var}(B)$. Purely combinatorial (computational), once $A$, $B$, and the diagram have been guessed.
How Gillibert proceeds for the critical point $\aleph_1$

- **Guess** a finite diagram, of finite distributive lattices and $(\lor, 0)$-homomorphisms:

\[
\begin{array}{c}
\{0, 1\}^2 \\
\{0, 1\}^2
\end{array}
\]

\[
\begin{array}{c}
\{0, 1\} \\
\{0, 1\}^4
\end{array}
\]

where $\varphi_1(x, y, z, t) := (x \lor y, z \lor t)$,

$\varphi_2(x, y, z, t) := (x \lor t, y \lor z)$,

$\psi(x, y) := x \lor y$. 

Prove that the diagram can be lifted in $\text{Var}(A)$, but not in $\text{Var}(B)$. Purely combinatorial (computational), once $A$, $B$, and the diagram have been guessed.
How Gillibert proceeds for the critical point $\aleph_1$

- **Guess** a finite diagram, of finite distributive lattices and $(\lor, 0)$-homomorphisms:

$$\begin{array}{c}
\{0, 1\} \\
\downarrow \psi \\
\{0, 1\}^2 \\
\downarrow \varphi_1 \\
\{0, 1\}^4 \\
\uparrow \psi \\
\{0, 1\}^2 \\
\uparrow \varphi_2
\end{array}$$

- Prove that the diagram can be lifted in $\text{Var}(A)$, but not in $\text{Var}(B)$. Purely combinatorial (computational), once $A$, $B$, and the diagram have been guessed.

where $\varphi_1(x, y, z, t) := (x \lor y, z \lor t)$, $\varphi_2(x, y, z, t) := (x \lor t, y \lor z)$, $\psi(x, y) := x \lor y$. 
How Gillibert concludes (critical point $\aleph_1$)

- **Prove** a “condensation principle”, that creates a “condensate” of the finite **diagram** above, which is a big **object** (algebraic distributive lattice with $\aleph_1$ compact elements).
How Gillibert concludes (critical point $\aleph_1$)

- Prove a “condensation principle”, that creates a “condensate” of the finite diagram above, which is a big object (algebraic distributive lattice with $\aleph_1$ compact elements).

- Any good (lifting) property of the big object (condensate) would be inherited by the small diagram. As the small diagram is bad, so is the big object.
How Gillibert concludes (critical point $\aleph_1$)

- **Prove** a “condensation principle”, that creates a “condensate” of the finite **diagram** above, which is a big **object** (algebraic distributive lattice with $\aleph_1$ compact elements).
- Any good (lifting) property of the big **object** (condensate) would be **inherited** by the small **diagram**. As the small diagram is bad, so is the big object.
- **Why $\aleph_1$?** This depends of the **shape** of the diagram (here, a square, $\{0,1\}^2$).

The "condensation principle" above has been subsequently set into a more general, categorical, framework.
How Gillibert concludes (critical point $\aleph_1$)

- **Prove a “condensation principle”,** that creates a “condensate” of the finite **diagram** above, which is a big **object** (algebraic distributive lattice with $\aleph_1$ compact elements).

- Any good (lifting) property of the big **object** (condensate) would be **inherited** by the small **diagram**. As the small diagram is bad, so is the big object.

- **Why $\aleph_1$?** This depends on the shape of the **diagram** (here, a **square**, $\{0,1\}^2$).

- The “condensation principle” above has been subsequently set into a more general, **categorical**, framework.
We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$. 
General categorical settings

We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$. We are trying to find a functor $\Gamma: \mathcal{A} \to \mathcal{B}$ such that $\Phi(A) \cong \Psi \Gamma(A)$, naturally in $\mathcal{A}$, for “many” (ideally, all) $A \in \mathcal{A}$. 
We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$. We are trying to find a functor $\Gamma: \mathcal{A} \to \mathcal{B}$ such that $\Phi(A) \cong \Psi \Gamma(A)$, naturally in $\mathcal{A}$, for “many” (ideally, all) $A \in \mathcal{A}$.

Hence we need an assumption of the form “for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$”.

Ask for $\Gamma: \mathcal{A} \to \mathcal{B}$ to be a functor (at least on a large enough subcategory of $\mathcal{A}$).
We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$. We are trying to find a functor $\Gamma: \mathcal{A} \to \mathcal{B}$ such that $\Phi(A) \cong \Psi \Gamma(A)$, naturally in $\mathcal{A}$, for "many" (ideally, all) $A \in \mathcal{A}$. 

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Phi} & \mathcal{S} \\
& \downarrow & \downarrow \Psi \\
& \mathcal{B} & \xrightarrow{\Phi} \mathcal{S} \\
\end{array}
\]

Hence we need an assumption of the form "for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$".
General categorical settings

We are given categories $\mathcal{A}$, $\mathcal{B}$, $S$ together with functors $\Phi: \mathcal{A} \to S$ and $\Psi: \mathcal{B} \to S$. We are trying to find a functor $\Gamma: \mathcal{A} \to \mathcal{B}$ such that $\Phi(A) \cong \Psi \Gamma(A)$, naturally in $\mathcal{A}$, for “many” (ideally, all) $A \in \mathcal{A}$.

Hence we need an assumption of the form “for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$”.

We are given categories $\mathcal{A}, \mathcal{B}, \mathcal{S}$ together with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$. We are trying to find a functor $\Gamma: \mathcal{A} \to \mathcal{B}$ such that $\Phi(A) \cong \Psi \Gamma(A)$, naturally in $\mathcal{A}$, for “many” (ideally, all) $A \in \mathcal{A}$.

Hence we need an assumption of the form “for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$”. Ask for $\Gamma: \mathcal{A} \rightarrow \mathcal{B}$ to be a functor (at least on a large enough subcategory of $\mathcal{A}$).
Larders

For an infinite regular cardinal $\lambda$, a $\lambda$-larder consists of categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$, together with a bunch of add-ons:
For an infinite regular cardinal $\lambda$, a $\lambda$-larder consists of categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ with functors $\Phi: \mathcal{A} \rightarrow \mathcal{S}$ and $\Psi: \mathcal{B} \rightarrow \mathcal{S}$, together with a bunch of add-ons:

- Full subcategories $\mathcal{A}^\uparrow \subseteq \mathcal{A}$, $\mathcal{B}^\uparrow \subseteq \mathcal{B}$ of “small” objects, plus a subcategory $\mathcal{S}^\Rightarrow \subseteq \mathcal{S}$ (the “double arrows”).
For an infinite regular cardinal $\lambda$, a $\lambda$-larder consists of categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ with functors $\Phi: \mathcal{A} \to \mathcal{S}$ and $\Psi: \mathcal{B} \to \mathcal{S}$, together with a bunch of add-ons:

- Full subcategories $\mathcal{A}^\dagger \subseteq \mathcal{A}$, $\mathcal{B}^\dagger \subseteq \mathcal{B}$ of “small” objects, plus a subcategory $\mathcal{S} \Leftrightarrow \subseteq \mathcal{S}$ (the “double arrows”).

- ... satisfying lots of extra properties (preservation properties related to colimits, plus an analogue of the Löwenheim-Skolem Theorem).
The statement of CLL is about as follows.
The statement of CLL is about as follows.

**Theorem (Gillibert and W., 2009)**

Let $\lambda$ be an infinite cardinal and let $P$ be a poset with a $\lambda$-lifter $(X, X^\lambda)$, let $(A, B, S, A^\dagger, B^\dagger, S \Rightarrow, \Phi, \Psi)$ be a $\lambda$-larder, let $\vec{A}$ be a $P$-indexed diagram in $A$ such that $A_p \in A^\dagger$ for each non-maximal $p \in P$, let $B \in B$ a $\lambda$-continuous directed colimit of a diagram in $B^\dagger$, and let $\chi : \Psi(\vec{B}) \Rightarrow \Phi(\Phi(F(X) \otimes \vec{A}))$. Then there are a $P$-indexed diagram $\vec{B}$ of subobjects of $B$ in $B^\dagger$ and a double arrow $\vec{\chi} : \Psi \vec{B} \Rightarrow \Phi \vec{A}$. In short: in order to lift the diagram $\Phi \vec{A}$ with respect to $\Psi \Rightarrow$, it is sufficient to lift the object $\Phi(A)$ with respect to $\Psi \Rightarrow$, where $A$ is a suitable condensate of $\vec{A}$ (viz. $A := F(X) \otimes \vec{A}$).
The statement of CLL is about as follows.

**Theorem (Gillibert and W., 2009)**

Let \( \lambda \) be an infinite cardinal and let \( P \) be a poset with a “\( \lambda \)-lifter” \((X, X)\), let \((\mathcal{A}, \mathcal{B}, S, \mathcal{A}^\dagger, \mathcal{B}^\dagger, S \Rightarrow, \Phi, \Psi)\) be a \( \lambda \)-larder, let \( \tilde{A} \) be a \( P \)-indexed diagram in \( \mathcal{A} \) such that \( A_p \in \mathcal{A}^\dagger \) for each non-maximal \( p \in P \), let \( B \in \mathcal{B} \) a \( \lambda \)-continuous directed colimit of a diagram in \( \mathcal{B}^\dagger \), and let \( \chi : \Psi(B) \Rightarrow \Phi(\mathbf{F}(X) \otimes \tilde{A}) \). Then there are a \( P \)-indexed diagram \( \tilde{B} \) of subobjects of \( B \) in \( \mathcal{B}^\dagger \) and a double arrow \( \chi : \Psi\tilde{B} \Rightarrow \Phi\tilde{A} \).
The Condensate Lifting Lemma (CLL)

The statement of CLL is about as follows.

**Theorem (Gillibert and W., 2009)**

Let $\lambda$ be an infinite cardinal and let $P$ be a poset with a “$\lambda$-lifter” $(X, \mathbf{X})$, let $(\mathcal{A}, \mathcal{B}, S, \mathcal{A}^\dagger, \mathcal{B}^\dagger, S\Rightarrow, \Phi, \Psi)$ be a $\lambda$-larder, let $\mathbf{A}$ be a $P$-indexed diagram in $\mathcal{A}$ such that $A_p \in \mathcal{A}^\dagger$ for each non-maximal $p \in P$, let $B \in \mathcal{B}$ a $\lambda$-continuous directed colimit of a diagram in $\mathcal{B}^\dagger$, and let $\chi : \Psi(B) \Rightarrow \Phi(\mathbf{F}(X) \otimes \mathbf{A})$. Then there are a $P$-indexed diagram $\mathbf{B}$ of subobjects of $B$ in $\mathcal{B}^\dagger$ and a double arrow $\chi : \Psi(\mathbf{B}) \Rightarrow \Phi(\mathbf{A})$.

In short: in order to lift the diagram $\Phi(\mathbf{A})$ with respect to $\Psi$, $\Rightarrow$, it is sufficient to lift the object $\Phi(A)$ with respect to $\Psi$, $\Rightarrow$, where $A$ is a suitable condensate of $\mathbf{A}$ (viz. $A := \mathbf{F}(X) \otimes \mathbf{A}$).
Limitations on the shape of $P$

- The poset $P$ in the statement of CLL needs to be an “almost join-semilattice with zero” (or a finite disjoint union of such guys).
Limitations on the shape of $P$

- The poset $P$ in the statement of CLL needs to be an "almost join-semilattice with zero" (or a finite disjoint union of such guys).
- In particular, CLL does not apply to diagrams indexed by the following posets:
Limitations on the shape of $P$

- The poset $P$ in the statement of CLL needs to be an “almost join-semilattice with zero” (or a finite disjoint union of such guys).
- In particular, CLL does not apply to diagrams indexed by the following posets:

  ![Posets](image)

- Too bad…
A ring (associative, not necessarily unital) $R$ is (von Neumann) **regular**, if $(\forall x \in R)(\exists y \in R)(xyx = x)$. 

Lattices of right ideals of von Neumann regular rings
Lattices of right ideals of von Neumann regular rings

- A ring (associative, not necessarily unital) \( R \) is (von Neumann) **regular**, if \((\forall x \in R)(\exists y \in R)(xyx = x)\).
- For a ring \( R \), set \( \mathbb{L}(R) := \{ xR \mid x \in R \} \).
Lattices of right ideals of von Neumann regular rings

- A ring (associative, not necessarily unital) $R$ is (von Neumann) regular, if $(\forall x \in R)(\exists y \in R)(xyx = x)$.
- For a ring $R$, set $\mathbb{L}(R) := \{xR \mid x \in R\}$.
- For $R := \mathbb{Z}[\sqrt{-5}]$, the poset $(\mathbb{L}(R), \subseteq)$ is not a lattice.
Lattices of right ideals of von Neumann regular rings

- A ring (associative, not necessarily unital) $R$ is (von Neumann) **regular**, if $(\forall x \in R)(\exists y \in R)(xyx = x)$.
- For a ring $R$, set $\mathbb{L}(R) := \{xR \mid x \in R\}$.
- For $R := \mathbb{Z}[\sqrt{-5}]$, the poset $(\mathbb{L}(R), \subseteq)$ is not a lattice.
- If $R$ is regular, then $\mathbb{L}(R)$ is a sectionally complemented sublattice of the right ideal lattice of $R$. In particular, it is modular (even Arguesian).
Lattices of right ideals of von Neumann regular rings

- A ring (associative, not necessarily unital) $R$ is (von Neumann) regular, if $(\forall x \in R)(\exists y \in R)(xyx = x)$.
- For a ring $R$, set $\mathbb{L}(R) : = \{ xR \mid x \in R \}$.
- For $R : = \mathbb{Z}[\sqrt{-5}]$, the poset $(\mathbb{L}(R), \subseteq)$ is not a lattice.
- If $R$ is regular, then $\mathbb{L}(R)$ is a sectionally complemented sublattice of the right ideal lattice of $R$. In particular, it is modular (even Arguesian).
- For a homomorphism $f : R \rightarrow S$ of regular rings, the map $\mathbb{L}(f) : \mathbb{L}(R) \rightarrow \mathbb{L}(S)$, $I \mapsto f(I)S$ is a 0-lattice homomorphism. The functor $\mathbb{L}$ thus defined preserves directed colimits (≡direct limits).
Latices of right ideals of von Neumann regular rings

- A ring (associative, not necessarily unital) $R$ is (von Neumann) regular, if $(\forall x \in R)(\exists y \in R)(xyx = x)$.
- For a ring $R$, set $\mathbb{L}(R) := \{xR \mid x \in R\}$.
- For $R := \mathbb{Z}[\sqrt{-5}]$, the poset $(\mathbb{L}(R), \subseteq)$ is not a lattice.
- If $R$ is regular, then $\mathbb{L}(R)$ is a sectionally complemented sublattice of the right ideal lattice of $R$. In particular, it is modular (even Arguesian).
- For a homomorphism $f : R \rightarrow S$ of regular rings, the map $\mathbb{L}(f) : \mathbb{L}(R) \rightarrow \mathbb{L}(S), I \mapsto f(I)S$ is a 0-lattice homomorphism. The functor $\mathbb{L}$ thus defined preserves directed colimits (\equiv direct limits).
- A lattice is coordinatizable, if it is isomorphic to $\mathbb{L}(R)$ for some regular ring $R$. 

---

Ladders and CLL
Lattices, congruences, varieties
Critical points between varieties
General settings; CLL
Coordinatization of lattices by regular rings
Non-coordinatizable SCMLs
Lattices without CPCP-extension
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)). \]
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)). \]

\( M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \} \), all \( a_i \) atoms, is 2-distributive.
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

$$x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)).$$

$$M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \},$$ all $a_i$ atoms, is 2-distributive.

A spanning $M_\omega$ in a bounded lattice $L$ is a 0, 1-sublattice of $L$ isomorphic to $M_\omega$. 

M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \}$, all $a_i$ atoms, is 2-distributive.

A spanning $M_\omega$ in a bounded lattice $L$ is a 0, 1-sublattice of $L$ isomorphic to $M_\omega$. 

Non-coordinatizable 2-distributive lattices

**The identity of 2-distributivity:**

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)) \]

\[ M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \} \], all \( a_i \) atoms, is 2-distributive. A spanning \( M_\omega \) in a bounded lattice \( L \) is a 0, 1-sublattice of \( L \) isomorphic to \( M_\omega \).

**Theorem (W., 2006)**

Every countable, 2-distributive complemented modular lattice with a spanning \( M_\omega \) is coordinatizable. The 0, 1-lattice embedding \( \phi: M_\omega \to M_\omega, a_n \mapsto a_{n+1} \) cannot be lifted with respect to the functor \( \mathcal{L} \). There exists a non-coordinatizable 2-distributive complemented modular lattice, of cardinality \( \aleph_1 \), with a spanning \( M_\omega \). In particular, coordinatizability is not first-order. (Established via a condensate-like construction)
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)). \]

\[ M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \}, \text{ all } a_i \text{ atoms, is 2-distributive.} \]

A spanning \( M_\omega \) in a bounded lattice \( L \) is a 0, 1-sublattice of \( L \) isomorphic to \( M_\omega \).

Theorem (W., 2006)

- Every countable, 2-distributive complemented modular lattice with a spanning \( M_\omega \) is coordinatizable.
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)). \]

Let \( M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \} \), all \( a_i \) atoms, be 2-distributive. A spanning \( M_\omega \) in a bounded lattice \( L \) is a 0, 1-sublattice of \( L \) isomorphic to \( M_\omega \).

**Theorem (W., 2006)**

- Every **countable**, 2-distributive complemented modular lattice with a spanning \( M_\omega \) is coordinatizable.
- The 0, 1-lattice embedding \( \varphi: M_\omega \hookrightarrow M_\omega, a_n \mapsto a_{n+1} \) cannot be lifted with respect to the functor \( \mathbb{I} \).
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)). \]

Let \( M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \} \), all \( a_i \) atoms, is 2-distributive.

A spanning \( M_\omega \) in a bounded lattice \( L \) is a 0, 1-sublattice of \( L \) isomorphic to \( M_\omega \).

**Theorem (W., 2006)**

- Every **countable**, 2-distributive complemented modular lattice with a spanning \( M_\omega \) is coordinatizable.
- The 0, 1-lattice embedding \( \varphi : M_\omega \hookrightarrow M_\omega, a_n \mapsto a_{n+1} \) cannot be lifted with respect to the functor \( \mathbb{L} \).
- There exists a non-coordinatizable 2-distributive complemented modular lattice, of cardinality \( \aleph_1 \), with a spanning \( M_\omega \).
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:
\[ x \wedge (y_0 \vee y_1 \vee y_2) = (x \wedge (y_0 \vee y_1)) \vee (x \wedge (y_0 \vee y_2)) \vee (x \wedge (y_1 \vee y_2)) \cdot \]

\[ M_\omega := \{0, 1, a_0, a_1, a_2, \ldots\} \text{, all } a_i \text{ atoms, is 2-distributive.} \]

A spanning \( M_\omega \) in a bounded lattice \( L \) is a 0, 1-sublattice of \( L \) isomorphic to \( M_\omega \).

Theorem (W., 2006)

- Every countable, 2-distributive complemented modular lattice with a spanning \( M_\omega \) is coordinatizable.
- The 0, 1-lattice embedding \( \varphi: M_\omega \hookrightarrow M_\omega, a_n \mapsto a_{n+1} \) cannot be lifted with respect to the functor \( \mathbb{I} \).
- There exists a non-coordinatizable 2-distributive complemented modular lattice, of cardinality \( \aleph_1 \), with a spanning \( M_\omega \). In particular, coordinatizability is not first-order. (Established via a condensate-like construction)
An element $a$ in a 0-lattice $L$ is large, if $\text{con}(0,a) = L \times L$. 
An element $a$ in a 0-lattice $L$ is large, if $\text{con}(0, a) = L \times L$. An $n$-frame in $L$ is a family $((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n})$ such that $(a_i)_{i < n}$ is independent and $c_i$ is an axis of perspectivity between $a_0$ and $a_i$ for each $i \in \{1, \ldots, n\}$.
An element \( a \) in a 0-lattice \( L \) is \textbf{large}, if \( \text{con}(0, a) = L \times L \). An \textbf{\( n \)-frame} in \( L \) is a family \( ((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n}) \) such that \( (a_i)_{i < n} \) is independent and \( c_i \) is an axis of perspectivity between \( a_0 \) and \( a_i \) for each \( i \in \{1, \ldots, n\} \). It is \textbf{large}, if \( a_0 \) is large.
Coordinatization of sectionally complemented modular lattices

An element $a$ in a 0-lattice $L$ is large, if $\text{con}(0, a) = L \times L$. An $n$-frame in $L$ is a family $((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n})$ such that $(a_i)_{i < n}$ is independent and $c_i$ is an axis of perspectivity between $a_0$ and $a_i$ for each $i \in \{1, \ldots, n\}$. It is large, if $a_0$ is large.

Theorem (Jónsson, 1962)
An element $a$ in a 0-lattice $L$ is large, if $\text{con}(0, a) = L \times L$. An $n$-frame in $L$ is a family $((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n})$ such that $(a_i)_{i < n}$ is independent and $c_i$ is an axis of perspectivity between $a_0$ and $a_i$ for each $i \in \{1, \ldots, n\}$. It is large, if $a_0$ is large.

**Theorem (Jónsson, 1962)**

Let $L$ be a sectionally complemented modular lattice with a large 4-frame.
An element $a$ in a 0-lattice $L$ is **large**, if $\text{con}(0, a) = L \times L$. An **$n$-frame** in $L$ is a family $((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n})$ such that $(a_i)_{i < n}$ is independent and $c_i$ is an axis of perspectivity between $a_0$ and $a_i$ for each $i \in \{1, \ldots, n\}$. It is **large**, if $a_0$ is large.

**Theorem (Jónsson, 1962)**

Let $L$ be a sectionally complemented modular lattice with a large 4-frame. If $L$ has a countable cofinal sequence, then $L$ is coordinatizable (i.e., $\exists R$ regular ring such that $L \cong \mathbb{L}(R)$).
Coordinatization of sectionally complemented modular lattices

An element $a$ in a 0-lattice $L$ is large, if $\text{con}(0, a) = L \times L$. An $n$-frame in $L$ is a family $((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n})$ such that $(a_i)_{i < n}$ is independent and $c_i$ is an axis of perspectivity between $a_0$ and $a_i$ for each $i \in \{1, \ldots, n\}$. It is large, if $a_0$ is large.

**Theorem (Jónsson, 1962)**

Let $L$ be a sectionally complemented modular lattice with a large 4-frame. If $L$ has a countable cofinal sequence, then $L$ is coordinatizable (i.e., $\exists R$ regular ring such that $L \cong \mathbb{L}(R)$).

**Theorem (W., 2008)**

There exists a non-coordinatizable sectionally complemented modular lattice, of cardinality $\aleph_1$, with a large 4-frame.
Coordinatization of sectionally complemented modular lattices

An element \( a \) in a 0-lattice \( L \) is large, if \( \text{con}(0, a) = L \times L \). An \( n \)-frame in \( L \) is a family \( ((a_i)_{0 \leq i < n}, (c_i)_{1 \leq i < n}) \) such that \( (a_i)_{i < n} \) is independent and \( c_i \) is an axis of perspectivity between \( a_0 \) and \( a_i \) for each \( i \in \{1, \ldots, n\} \). It is large, if \( a_0 \) is large.

**Theorem (Jónsson, 1962)**

Let \( L \) be a sectionally complemented modular lattice with a large 4-frame. If \( L \) has a countable cofinal sequence, then \( L \) is coordinatizable (i.e., \( \exists R \) regular ring such that \( L \cong \mathbb{L}(R) \)).

**Theorem (W., 2008)**

There exists a non-coordinatizable sectionally complemented modular lattice, of cardinality \( \aleph_1 \), with a large 4-frame.
Why larders there?

Larders don’t play any role in the proof of the latter result, until we reach a $\omega_1$-tower of sectionally complemented modular lattices that cannot be lifted by the $\mathbb{L}$ functor.
Why larders there?

- Larders don’t play any role in the proof of the latter result, until we reach a $\omega_1$-tower of sectionally complemented modular lattices that cannot be lifted by the $\mathbb{L}$ functor.
- Then larders are used to turn the diagram counterexample to an object counterexample.
Lattices without congruence-permutable, congruence-preserving extension

An extension $A \leq B$ of (universal) algebras is **congruence-preserving**, if the canonical map $\text{Con} \ A \rightarrow \text{Con} \ B$ is an isomorphism.
Lattices without congruence-permutable, congruence-preserving extension

An extension $A \leq B$ of (universal) algebras is congruence-preserving, if the canonical map $\text{Con} A \rightarrow \text{Con} B$ is an isomorphism.

Theorem (Gillibert and W., 2009)

Due to earlier results of Ploščica, Tůma, and W., the analogue of this result at $\aleph_2$ was already known. Furthermore, if $V$ is locally finite, then $\aleph_1$ is optimal in the result above. (Open problem in the non locally finite case. For example: does the free lattice on $\aleph_0$ generators have a congruence-permutable, congruence-preserving extension?). Unlike all previous examples, the larder data for this result are difficult to figure out.
Lattices without congruence-permutable, congruence-preserving extension

An extension $A \leq B$ of (universal) algebras is **congruence-preserving**, if the canonical map $\text{Con } A \rightarrow \text{Con } B$ is an isomorphism.

**Theorem (Gillibert and W., 2009)**

Let $\mathcal{V}$ be a nondistributive lattice variety. Then the free lattice (resp., the free bounded lattice) on $\aleph_1$ generators within $\mathcal{V}$ has no congruence-permutable, congruence-preserving extension.
Lattices without congruence-permutable, congruence-preserving extension

An extension $A \leq B$ of (universal) algebras is **congruence-preserving**, if the canonical map $\text{Con } A \to \text{Con } B$ is an isomorphism.

**Theorem (Gillibert and W., 2009)**

Let $\mathcal{V}$ be a nondistributive lattice variety. Then the free lattice (resp., the free bounded lattice) on $\aleph_1$ generators within $\mathcal{V}$ has no congruence-permutable, congruence-preserving extension.

Due to earlier results of Ploščica, Tůma, and W., the analogue of this result at $\aleph_2$ was already known.
Lattices without congruence-permutable, congruence-preserving extension

An extension $A \leq B$ of (universal) algebras is **congruence-preserving**, if the canonical map $\text{Con } A \to \text{Con } B$ is an isomorphism.

**Theorem (Gillibert and W., 2009)**

Let $\mathcal{V}$ be a nondistributive lattice variety. Then the free lattice (resp., the free bounded lattice) on $\aleph_1$ generators within $\mathcal{V}$ has no congruence-permutable, congruence-preserving extension.

Due to earlier results of Ploščica, Tůma, and W., the analogue of this result at $\aleph_2$ was already known. Furthermore, if $\mathcal{V}$ is locally finite, then $\aleph_1$ is optimal in the result above. (Open problem in the non locally finite case. For example: does the free lattice on $\aleph_0$ generators have a congruence-permutable, congruence-preserving extension?).
Lattices without congruence-permutable, congruence-preserving extension

An extension $A \leq B$ of (universal) algebras is congruence-preserving, if the canonical map $\text{Con} A \rightarrow \text{Con} B$ is an isomorphism.

**Theorem (Gillibert and W., 2009)**

Let $\mathcal{V}$ be a nondistributive lattice variety. Then the free lattice (resp., the free bounded lattice) on $\aleph_1$ generators within $\mathcal{V}$ has no congruence-permutable, congruence-preserving extension.

Due to earlier results of Ploščica, Tůma, and W., the analogue of this result at $\aleph_2$ was already known. Furthermore, if $\mathcal{V}$ is locally finite, then $\aleph_1$ is optimal in the result above. (Open problem in the non locally finite case. For example: does the free lattice on $\aleph_0$ generators have a congruence-permutable, congruence-preserving extension?).

Unlike all previous examples, the larder data for this result are difficult to figure out.