From lifting objects to lifting diagrams: recent progress on larders and CLL

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Most of the results discussed here obtained with Pierre Gillibert.

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General categorical settings

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Hence we need an assumption of the form “for many $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\Phi(A) \cong \Psi(B)$”.

\[ \begin{array}{ccc}
\Phi & \longrightarrow & \Psi \\
\mathcal{A} & \quad & \mathcal{B} \\
\downarrow \Phi & & \downarrow \Psi \\
\mathcal{S} & & \mathcal{S} \\
\Gamma & \longrightarrow & \\
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Let’s see some examples.
Theorem (Schmidt 1981)

For each distributive 0-lattice $D$, there exists a lattice $L$ such that $\text{Con}_{L}$, the $(\lor, 0)$-semilattice of all compact (finitely generated) congruences of $L$, is isomorphic to $D$.
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- $\Psi: \mathcal{B} \to \mathcal{S}$, $L \mapsto \text{Con}_c L$ (naturally extended to homomorphisms).
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In fact, the functor \( \Gamma \) constructed in Pudlák's proof sends finite distributive lattices to finite atomistic lattices, and preserves directed colimits (\( = \) direct limits).
Distributive 0-lattices as compact congruence semilattices of lattices (...but not too functorially)

Question (Pudlák 1985)

Can this be done with \( \Gamma: (\text{distr. 0-semilatt.}, (\lor, 0))-\text{embeddings} \rightarrow (\text{latt.}, \text{latt. emb.}) \)?

(\text{Note: there is no hope with } \Gamma: (\text{distr. 0-semilatt.}, (\lor, 0))-\text{homomorphisms} \rightarrow (\text{latt.}, \text{latt. hom.}), \text{for "trivial" reasons.})

\text{Answer (T˚ uma and W., 2006)}

No, it cannot. (For nontrivial reasons, that can be extended to any variety with a nontrivial congruence \( (\lor, \land) \)-identity.)
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Distributive 0-lattices as compact ideal semilattices of locally matricial algebras (at object level)

An algebra $R$ over a field $F$ is

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- **General settings**
- **P-scaled algebras**
- **Lifters, larders, and CLL**
- **Diagram form of GS**
- **Relative critical points**
- **Non-coordinatizable SCMLs**
- **Lattices without CPCP-extension**
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Due to the link between K-theory of regular rings and congruence lattices of lattices, Růžička's result extends Schmidt's result.
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Critical points between varieties of algebras

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Let $\mathcal{A}$ be a locally finite variety and let $\mathcal{B}$ be a finitely generated congruence-distributive variety. Then $\text{Con}_c \mathcal{A} \not\subseteq \text{Con}_c \mathcal{B}$ implies that $\text{crit}(\mathcal{A}; \mathcal{B}) < \aleph_\omega$. 
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Whether all $\aleph_n$ can be thus reached (for finite similarity types) is a difficult open problem. (However, some partial results are known.)
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Larders and CLL

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For a homomorphism \( f : R \to S \) of regular rings, the map \( \mathbb{L}(f) : \mathbb{L}(R) \to \mathbb{L}(S), I \mapsto f(I)S \) is a 0-lattice homomorphism. The functor \( \mathbb{L} \) thus defined preserves directed colimits (=direct limits).
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- A lattice is **coordinatizable**, if it is isomorphic to $\mathbb{L}(R)$ for some regular ring $R$. 
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)). \]
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Let \( M_\omega := \{0, 1, a_0, a_1, a_2, \ldots \} \), all \( a_i \) atoms, be 2-distributive. A spanning \( M_\omega \) in a bounded lattice \( L \) is a 0, 1-sublattice of \( L \) isomorphic to \( M_\omega \).

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- Every **countable**, 2-distributive complemented modular lattice with a spanning \( M_\omega \) is coordinatizable.
Non-coordinatizable 2-distributive lattices

The identity of 2-distributivity:

\[ x \land (y_0 \lor y_1 \lor y_2) = (x \land (y_0 \lor y_1)) \lor (x \land (y_0 \lor y_2)) \lor (x \land (y_1 \lor y_2)) . \]

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Non-coordinatizable 2-distributive lattices

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- There exists a non-coordinatizable 2-distributive complemented modular lattice, of cardinality \( \aleph_1 \), with a spanning \( M_\omega \). In particular, coordinatizability is not first-order.
An **ideal** of a poset $P$ is a nonempty, upward directed lower subset of $P$. Denote by $\text{Id} P$ the set of all ideals of $P$, ordered by containment.
**$P$-normed topological spaces**

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**Larders and CLL**

**General settings**

**$P$-scaled algebras**

**Lifters, larders, and CLL**

**Diagram form of GS**

**Relative critical points**

**Non-coordinatizable SCMLs**

**Lattices without CPCP-extension**
A \textbf{P-normed (topological) space} is a pair $X = (X, \nu)$, where $X$ is a topological space, $\nu: X \rightarrow \text{Id } P$, and the subset $\{x \in X \mid p \in \nu(x)\}$ is open in $X$, for each $p \in P$. 

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- Write $\|x\|$, or $\|x\|_X$, instead of $\nu(x)$. 

- **BTop** $P$ := category of all $P$-normed Boolean spaces with morphisms as above.

A description of the dual category follows.
A **$P$-normed (topological) space** is a pair $X = (X, \nu)$, where $X$ is a topological space, $\nu: X \to \text{Id} \, P$, and the subset $\{x \in X \mid p \in \nu(x)\}$ is open in $X$, for each $p \in P$.

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- For $P$-normed spaces $X$ and $Y$, a **morphism** $X \to Y$ is a continuous map $f: X \to Y$ such that $\|f(x)\|_Y \subseteq \|x\|_X$ for each $x \in X$. 

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Fix a poset \( P \).
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**Definition (Gillibert and W., 2009)**

A $P$-scaled Boolean algebra is a structure $A = (A, (A(p)|p \in P))$, where $A$ is a Boolean algebra, each $A(p)$ is an ideal of $A$, and $\bigvee (A(p)|p \in P)$ in $\text{Id} A$; $A(p) \cap A(q) = \bigvee (A(r)|r \geq p, q)$ for all $p, q \in P$.

For $P$-scaled Boolean algebras $A$ and $B$, a morphism from $A$ to $B$ is a homomorphism $f: A \rightarrow B$ of Boolean algebras such that $f(A(p)) \subseteq B(p)$ for each $p \in P$.

Denote by $\text{Bool}_P$ the category of all $P$-scaled Boolean algebras with above described morphisms.
$P$-scaled Boolean algebras

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**$P$-scaled Boolean algebras**
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- Denote by $\text{Bool}_P$ the category of all $P$-scaled Boolean algebras with above described morphisms.
For a $P$-scaled Boolean algebra $A$, we set
\[
\|a\| := \{ p \in P \mid a \cap A(p) \neq \emptyset \}, \quad \text{for each } a \in \text{Ult } A.
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Duality between $\mathbf{BTop}_P$ and $\mathbf{Bool}_P$

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- The structure $\text{Clop } X := (A, (A^{(p)} \mid p \in P))$ is a $P$-scaled Boolean algebra.
Let $P$ be a poset.
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**Proposition (Gillibert and W., 2009)**

The pair $(\text{Ult}_P, \text{Clop})$ defines a duality between the category $\text{BTop}_P$ of all $P$-normed Boolean spaces and the category $\text{Bool}_P$ of all $P$-scaled Boolean algebras.

**Proposition (Gillibert and W., 2009)**

The category $\text{Bool}_P$ has all nonempty small directed colimits.

The category $\text{Bool}_P$ has all nonempty finite products. Furthermore, if $P$ is finite, then $\text{Bool}_P$ has all nonempty small products.
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Basic categorical properties of $\mathbf{BTop}_P$ and $\mathbf{Bool}_P$

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Finitely presented objects in a category

Definition (Gabriel and Ulmer 1971)

An object $A$ in a category $C$ is finitely presented, if for every directed colimit representation $(X_i, x_{i|j})_{i \in I} = \lim_{\rightarrow} (X_i, x_{j|i})_{i \leq j \in I}$ in $C$, $\forall f: A \to X_i$, $\exists i \in I$ such that $f$ factors through $X_i$; $\forall i \in I$ and $\forall f, g: A \to X_i$, $x_{i|i} \circ f = x_{i|i} \circ g \Rightarrow (\exists j \geq i) (x_{j|i} \circ f = x_{j|i} \circ g)$.

For example, an element in a poset is finitely presented iff it is compact.
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Finitely presented objects in $\text{Bool}_P$

Proposition (Gillibert and W., 2009)

A $P$-scaled Boolean algebra $A$ is finitely presented in $\text{Bool}_P$ if and only if $A$ is finite and $\|a\|$ is a principal ideal for each ultrafilter $a$ of $A$.

Every $P$-scaled Boolean algebra is a monomorphic directed colimit of finitely presented $P$-scaled Boolean algebras.
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Normal morphisms of $P$-scaled Boolean algebras

Definition (Gillibert and W., 2009)

A morphism $f: A \rightarrow B$ of $P$-scaled Boolean algebras is normal, if it is surjective and $f(A(p)) = B(p)$ for each $p \in P$.

It is compact, if both $A$ and $B$ are finitely presented.

For an ideal $I$ of $A$, one can define a $P$-scaled Boolean algebra $A/I$ of underlying algebra $A/I$, with $(A/I)(p) = A(p)/I$ for each $p \in P$.

The projection map $A \rightarrow A/I$ is a normal morphism, and every normal morphism has this form (up to isomorphism).

The normal morphisms of $\text{Bool}^P$ are exactly its regular epimorphisms (i.e., coequalizers of two morphisms).

Proposition (Gillibert and W., 2009)

Every normal morphism in $\text{Bool}^P$ is a directed colimit of compact normal morphisms.
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Defining $A \otimes \vec{S}$ for $A$ finitely presented

- Work in a category $S$ with all nonempty finite products, and fix a poset $P$. 

Let $\vec{S} = (S_p, \sigma_q | p \leq q \text{ in } P)$ be a $P$-indexed diagram in $S$.

Let $A$ be a finitely presented $P$-scaled Boolean algebra.

For each atom $u$ of $A$, denote by $|u|$ the largest $p \in P$ such that $u \in A(p)$.

Set $A \otimes \vec{S} := \prod (S_{|u|} | u \in \text{At}(A))$.

For a morphism $\phi : A \rightarrow B$ in $\text{Bool}_P$, one can define naturally a morphism $\phi \otimes \vec{S} : A \otimes \vec{S} \rightarrow B \otimes \vec{S}$ in $S$.

We get a $S$-valued functor $A \mapsto A \otimes \vec{S}$, defined on the finitely presented part of $\text{Bool}_P$. 
Defining $A \otimes \vec{S}$ for $A$ finitely presented

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\[ A \otimes \vec{S} := \prod (S_p, \sigma^q_p | u \in \text{At } A) \] 

For a morphism $\phi: A \rightarrow B$ in $\text{Bool}_P$, one can define naturally a morphism $\phi \otimes \vec{S}: A \otimes \vec{S} \rightarrow B \otimes \vec{S}$ in $S$.

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Lattices without CPCP-extension
Defining $A \otimes \vec{S}$ for $A$ finitely presented

- Work in a category $\mathcal{S}$ with all nonempty finite products, and fix a poset $P$.
- Let $\vec{S} = (S_p, \sigma_p^q \mid p \leq q \text{ in } P)$ be a $P$-indexed diagram in $\mathcal{S}$.
- Let $A$ be a finitely presented $P$-scaled Boolean algebra. For each atom $u$ of $A$, denote by $|u|$ the largest $p \in P$ such that $u \in A(p)$.
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- We get a $S$-valued functor $A \mapsto A \otimes \vec{S}$, defined on the finitely presented part of $\text{Bool}_P$. 
Defining $A \otimes \vec{S}$ in general

Let $\mathcal{S}$ be a category with all nonempty finite products and all nonempty small directed colimits, and let $\vec{S}$ be a $P$-indexed diagram in $\mathcal{S}$.
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**Proposition (Gillibert and W., 2009)**

The functor $\mathbf{A} \mapsto \mathbf{A} \otimes \vec{S}$ can be uniquely (up to iso) extended to a directed colimits preserving functor from $\mathsf{Bool} P$ to $S$. This way, $\mathbf{A} \otimes \vec{S}$ is defined for any $\mathbf{A} \in \mathsf{Bool} P$. Also $\phi \otimes \vec{S}$, for $\phi : \mathbf{A} \to \mathbf{B}$ in $\mathsf{Bool} P$. We say that $\mathbf{A} \otimes \vec{S}$ is a condensate of $\vec{S}$.

A projection in $S$ is either an isomorphism or a factor morphism $X \times Y \to X$ in $S$. An extended projection is a directed colimit of projections (in $S$).

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If a morphism $\phi : \mathbf{A} \to \mathbf{B}$ in $\mathsf{Bool} P$ is normal, then $\phi \otimes \vec{S}$ is an extended projection in $S$. 
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Special sorts of posets

For a subset $X$ in a poset $P$, we set

$$P \uparrow X := \{ p \in P \mid X \leq p \}$$

and

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- $P$ is a pseudo join-semilattice, if $P \uparrow X$ is a finitely generated upper subset of $P$, for any finite $X \subseteq P$. 
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- $P$ is an almost join-semilattice, if it is a pseudo join-semilattice and $P \downarrow a$ is a join-semilattice $\forall a \in P$. 
Special sorts of posets

- For a subset \( X \) in a poset \( P \), we set
  \[ P \uparrow X := \{ p \in P \mid X \leq p \} \text{ and } \triangledown X := \text{Min}(P \uparrow X). \]
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- \( P \) is a **pseudo join-semilattice**, if \( P \uparrow X \) is a finitely generated upper subset of \( P \), for any finite \( X \subseteq P \).
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- \( P \) is an **almost join-semilattice**, if it is a pseudo join-semilattice and \( P \downarrow a \) is a join-semilattice \( \forall a \in P \).
- \((\text{pseudo join-semilattice}) \Rightarrow (\text{supported}) \Rightarrow (\text{almost join-semilattice})\); the converses do not hold.
Norm-coverings and $\lambda$-lifters

- A **norm-covering** of a poset $P$ is a pair $(X, \partial)$, where $X$ is a pseudo join-semilattice and $\partial: X \rightarrow P$ is isotone.
Norm-coverings and $\lambda$-lifters

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Let $\lambda$ be an infinite cardinal. A $\lambda$-lifter of $P$ is a pair $(X, X)$, where $X$ is a norm-covering of $P$, $X$ is a set of sharp ideals of $X$, and, setting $X^= := \{ x \in X \mid \partial x \text{ not maximal} \}$,

1. $\text{card}(X \downarrow x) < \lambda$ for each $x \in X^=$.
2. (Kuratowski-like property) For each isotone $S : X^= \to [X]^{<\lambda}$, there exists an isotone $\sigma : P \to X$ such that
   1. $\partial \circ \sigma = \text{id}_P$;
   2. $\forall p < q \in P, S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$.
3. If $\lambda = \aleph_0$, then $X$ is supported.
The $P$-scaled Boolean algebras $\mathbf{F}(X)$

For a norm-covering $\partial: X \rightarrow P$, denote by $\mathbf{F}(X)$ the Boolean algebra defined by generators $\tilde{u}$ (for $u \in X$) and relations

1. $\tilde{v} \leq \tilde{u}$, for all $u \leq v$ in $X$;
2. $\tilde{u} \wedge \tilde{v} = \bigvee (\tilde{w} \mid w \in u \triangleleft v)$, for all $u, v \in X$;
3. $1 = \bigvee (\tilde{u} \mid u \in \text{Min } X)$. 


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- Then define $F(X)^{(p)}$ as the ideal of $F(X)$ generated by $\{\tilde{u} \mid u \in X$ and $p \leq \partial u\}$, for each $p \in P$. 
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- The assignment $X \mapsto F(X)$ has nice functorial properties.
More on $\lambda$-lifters

Proposition (Gillibert and W., 2009)

If a poset $P$ has a $\lambda$-lifter, then $P$ is a finite disjoint union of almost join-semilattices with zero (in particular, it is an almost join-semilattice). Every finite almost join-semilattice $P$ has a $\lambda$-lifter ($\lambda$ arbitrary infinite cardinal). The minimal cardinality of a possible underlying $X$ is $\leq \lambda + (\dim P - 1)$ (and $< \lambda$ may occur). For infinite $P$, the existence of $\lambda$-lifters is related to large cardinal axioms, for instance Erdős cardinals.
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Moving to the definition of a $\lambda$-larder

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- We are given categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ together with functors $\Phi : \mathcal{A} \to \mathcal{S}$ and $\Psi : \mathcal{B} \to \mathcal{S}$.

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\downarrow \\
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![Diagram]

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An octuple $(\mathcal{A}, \mathcal{B}, S, \mathcal{A}^\dagger, \mathcal{B}^\dagger, S\Rightarrow, \Phi, \Psi)$ is a $\lambda$-larder, if $\mathcal{A}^\dagger \subseteq \mathcal{A}$ full, $\mathcal{B}^\dagger \subseteq \mathcal{B}$ full, $S\Rightarrow \subseteq S$ subcategory, $B \in \mathcal{B}^\dagger$ is $\lambda$-presented in $\mathcal{B}$ and $\Psi(B)$ is $\lambda$-presented in $S$ for each $B \in \mathcal{B}^\dagger$, $\mathcal{A}$ has all nonempty small directed limits and all nonempty finite products, $S\Rightarrow$ is “closed under nonempty small directed limits”, $\Phi$ preserves nonempty small directed limits, $\Psi$ preserves nonempty $\lambda$-small directed limits, $\Phi$(projections) $\subseteq S\Rightarrow$, and (Löwenheim-Skolem Property) for each $S \in \Phi(\mathcal{A}^\dagger)$, each $B \in \mathcal{B}$, and each $\varphi : \Psi(B) \to S$ in $S\Rightarrow$ there are “many” $u : U \to B$ with $U \in \mathcal{B}^\dagger$ such that $\varphi \circ \Psi(u) \in S\Rightarrow$. 
The double arrows

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- For example, if $S$ is the category of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms, $S \Rightarrow$ is often the subcategory with morphisms of the form $S \to S/I$ ($I$ ideal of $S$) up to iso, and then any double arrow $\varphi : \text{Con}_c U \Rightarrow S$ can be “nicely factored” through an isomorphism.
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The Condensate Lifting Lemma (CLL)

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**Theorem (Gillibert and W., 2009)**

Let $\lambda$ be an infinite cardinal and let $P$ be a poset with a $\lambda$-lifter $(X, X)$, let $(A, B, S, A^\dagger, B^\dagger, S \Rightarrow, \Phi, \Psi)$ be a $\lambda$-larder, let $\vec{A}$ be a $P$-indexed diagram in $A$ such that $A_p \in A^\dagger$ for each non-maximal $p \in P$, let $B \in B$ a $\lambda$-continuous directed colimit of a diagram in $B^\dagger$, and let $\chi: \Psi(\vec{B}) \Rightarrow \Phi(\Phi(\vec{A}) \otimes \vec{A})$. Then there are a $P$-indexed diagram $\vec{B}$ of subobjects of $B$ in $B^\dagger$ and a double arrow $\vec{\chi}: \Psi \vec{B} \Rightarrow \Phi \vec{A}$. In short: in order to lift the diagram $\Phi \vec{A}$ with respect to $\Psi \Rightarrow$, it is sufficient to lift the object $\Phi(A)$ with respect to $\Psi \Rightarrow$, where $A$ is a suitable condensate of $\vec{A}$ (viz. $A := F(X) \otimes \vec{A}$).
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In short: in order to lift the diagram $\Phi \vec{A}$ with respect to $\Psi$, $\Rightarrow$, it is sufficient to lift the object $\Phi(A)$ with respect to $\Psi$, $\Rightarrow$, where $A$ is a suitable condensate of $\vec{A}$ (viz. $A := F(X) \otimes \vec{A}$).
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- The poset $P$ in the statement of CLL needs to be an almost join-semilattice with zero (or a finite disjoint union of such guys).
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- The poset $P$ in the statement of CLL needs to be an almost join-semilattice with zero (or a finite disjoint union of such guys).
- In particular, CLL does not apply to diagrams indexed by the following posets:
Limitations on the shape of $P$

- The poset $P$ in the statement of CLL needs to be an almost join-semilattice with zero (or a finite disjoint union of such guys).
- In particular, CLL does not apply to diagrams indexed by the following posets:

  ![Diagram of posets](image)

- Too bad…
The Grätzer-Schmidt Theorem

Theorem (Grätzer and Schmidt, 1963)

Every \((\lor, 0)\)-semilattice is isomorphic to \(\text{Con}_c A\), for some (universal) algebra \(A\). Of course \(A\) can be unary. Nevertheless, due to a 1979 paper by Freese, Lampe, and Taylor, there is no bound on the cardinality of the similarity type of the algebra \(A\).

Hence, if we want to state a diagram version of the GS Theorem, we need to work in a suitable category of non-indexed algebras.

Among 3 possible definitions of non-indexed algebras, 2 of them won't satisfy the assumptions of CLL. The one that works is the following: consider the category \(\text{MAlg}_1\) of all unary algebras, where \(f: A \to B\) means that \(\text{Op}(A) \subseteq \text{Op}(B)\) and \(f\) is a homomorphism for all symbols in \(\text{Op}(A)\).
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The Grätzer-Schmidt Theorem (introducing the larder data)

- Denote by $\text{Sem}_{\lor,0}$ the category of all $(\lor, 0)$-semilattices with $(\lor, 0)$-homomorphisms.
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- For an infinite regular cardinal $\lambda$, denote by $\mathbf{Sem}_{\lor, 0}^{(\lambda)}$ the class of all $(\lor, 0)$-semilattices of cardinality $<\lambda$. Similarly for $\mathbf{MAlg}_{1}^{(\lambda)}$ (require $\text{card } A + \text{card } \text{Op}(A) < \lambda$).
The Grätzer-Schmidt Theorem (picture of the larder data)

\[ S \Rightarrow := \text{ideal-ind. homs} \]

\[ S := \text{Sem}_{\vee,0} \]

\[ \Phi := \text{id} \]

\[ \Psi := \text{Con}_c \]

\[ A \Rightarrow := \text{ideal-ind. homs} \]

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\[ A^\dagger := \text{Sem}^{(\lambda)}_{\vee,0} \]

\[ B \Rightarrow := \text{MAlg}_1 \]

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The Grätzer-Schmidt Theorem (diagram version)

Theorem (Gillibert and W., 2009)

Let $P$ be a poset and let $\vec{S}$ be a $P$-indexed diagram of $(\lor, 0)$-semilattices and $(\lor, 0)$-homomorphisms. If either $P$ is finite, or $P$ is infinite and "a large enough cardinal exists", then $\vec{S}$ has a lifting, wrt. the $\text{Con}_c$ functor, by a diagram of unary algebras and homomorphisms in $\text{MAlg}^1$. The large cardinal axiom in question states the existence of large independent sets for certain set functions (cf. Kuratowski's Free Set Theorem). If there is a proper class of Erdős cardinals (this axiom is weaker, consistency-wise, than a Ramsey cardinal), then this assumption is satisfied for any poset $P$. 
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Relative critical points between quasivarieties

- **Quasivariety** of structures: class of first-order structures, in a given first-order language, closed under $\mathbf{S}$, $\mathbf{P}$, and directed limits.
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- For a structure $A$ and a quasivariety $\mathcal{V}$ (in the same language), set $\text{Con}^\mathcal{V}A := \{ \alpha \in \text{Con}A \mid A/\alpha \in \mathcal{V} \}$. 

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- Then set $\text{Con}_{c,r}^\mathcal{V} := \{ S \in \text{Sem}^\mathcal{V,0} \mid (\exists A \in \mathcal{V})(S \cong \text{Con}^\mathcal{V}A) \}$. 
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- For a structure $A$ and a quasivariety $\mathcal{V}$ (in the same language), set $\text{Con}^{\mathcal{V}} A := \{ \alpha \in \text{Con} A \mid A/\alpha \in \mathcal{V} \}$. In particular, $\text{Con}^{\mathcal{V}} A$ is an algebraic lattice.

- Then set

  \[ \text{Con}_{c,r}^{\mathcal{V}} := \{ S \in \text{Sem}_{\mathcal{V},0} \mid (\exists A \in \mathcal{V})(S \cong \text{Con}_{c}^{\mathcal{V}} A) \} \]

- For quasivarieties $\mathcal{A}$ and $\mathcal{B}$ (not necessarily in the same language), set

  \[ \text{crit}_{r}(\mathcal{A}; \mathcal{B}) := \min\{ \text{card } S \mid S \in (\text{Con}_{c,r}^{\mathcal{A}}) \setminus (\text{Con}_{c,r}^{\mathcal{B}}) \} \]

  if it exists, $\infty$ otherwise.
Description of the larder data

Small variations around the following:

\[ S \Rightarrow := \text{ideal-ind. homs} \]
\[ S := \text{Sem}_{\lor,0} \]

\[ \Phi := \text{Con}_c^A \]
\[ \Psi := \text{Con}_c^B \]

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Upper bounds for relative critical points

Theorem (Gillibert and W., 2009)

Let $A$ and $B$ be quasivarieties (possibly in different languages), such that the language of $A$ has only finitely many relations and $B$ is finitely generated (no need for CD), and let $P$ be a nontrivial finite almost join-semilattice with zero. If there exists a $P$-indexed diagram $\vec{A}$ of objects of $A$ with finite universe such that $\text{Con}_A c \vec{A}$ has no lifting, wrt. $\text{Con}_B c$, in $B$, then $\text{crit}_{r}(A; B) \leq \aleph_0 \text{dim}(P) - 1$.

Furthermore, $\text{Con} c \vec{A} \not\subseteq \text{Con} c \vec{B}$ implies that $\text{crit}_{r}(A; B) < \aleph_\omega$.

(First obtained for varieties by Gillibert)

Here, $\text{dim}(P)$ denotes the order-dimension of $P$. The inequality $\text{crit}_{r}(A; B) < \aleph_0 \text{dim}(P) - 1$ may hold.
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- Furthermore, $\text{Con}_{c,r}^{\mathcal{A}} \nsubseteq \text{Con}_{c,r}^{\mathcal{B}}$ implies that $\text{crit}_{r}(\mathcal{A}; \mathcal{B}) < \aleph_{\omega}$. 
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- Furthermore, $\text{Con}_{c,r} \mathcal{A} \not\subseteq \text{Con}_{c,r} \mathcal{B}$ implies that $\text{crit}_r(\mathcal{A}; \mathcal{B}) < \aleph_\omega$. (First obtained for varieties by Gillibert)
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- Furthermore, $\text{Con}^\mathcal{A}_{c,r} \not\subseteq \text{Con}^\mathcal{B}_{c,r}$ implies that $\text{crit}_r(\mathcal{A}; \mathcal{B}) < \aleph_\omega$. (First obtained for varieties by Gillibert)

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- Let $\mathcal{A}$ and $\mathcal{B}$ be quasivarieties (possibly in different languages), such that the language of $\mathcal{A}$ has only finitely many relations and $\mathcal{B}$ is finitely generated (*no need for CD*), and let $P$ be a nontrivial finite almost join-semilattice with zero. If there exists a $P$-indexed diagram $\vec{A}$ of objects of $\mathcal{A}$ with finite universe such that $\text{Con}^\mathcal{A}_c \vec{A}$ has no lifting, wrt. $\text{Con}^\mathcal{B}_c$, in $\mathcal{B}$, then $\text{crit}_r(\mathcal{A}; \mathcal{B}) \leq \aleph_{\text{dim}(P)-1}$.
- Furthermore, $\text{Con}^\mathcal{A}_{c,r} \not\subseteq \text{Con}^\mathcal{B}_{c,r}$ implies that $\text{crit}_r(\mathcal{A}; \mathcal{B}) < \aleph_\omega$. (*First obtained for varieties by Gillibert*)

- Here, $\text{dim}(P)$ denotes the *order-dimension* of $P$.
- The inequality $\text{crit}_r(\mathcal{A}; \mathcal{B}) < \aleph_{\text{dim}(P)-1}$ may hold.
Actually, \( \text{crit}_r(\mathcal{A}; \mathcal{B}) \leq \aleph_{\text{kur}_0(P) - 1} \), where \( \text{kur}_0(P) \), the "restricted Kuratowski index of \( P \)" , is the least positive integer \( n \) such that a certain "existence of large independent sets"-type statement, denoted by \((\aleph_{n-1}, <\aleph_0) \rightharpoonup P\), holds.
Actually, $\text{crit}_r(\mathcal{A}; \mathcal{B}) \leq \aleph_{\text{kur}_0(P) - 1}$, where $\text{kur}_0(P)$, the “restricted Kuratowski index of $P$”, is the least positive integer $n$ such that a certain “existence of large independent sets”-type statement, denoted by $(\aleph_{n-1}, < \aleph_0) \simeq P$, holds. In particular, $\text{kur}_0(P) \leq \dim(P)$. 

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Restricted Kuratowski index of a finite poset
Actually, \( \text{crit}_r(\mathcal{A}; \mathcal{B}) \leq \aleph_{\text{kur}_0(P)-1} \), where \( \text{kur}_0(P) \), the “restricted Kuratowski index of \( P \)”, is the least positive integer \( n \) such that a certain “existence of large independent sets”-type statement, denoted by \( (\aleph_{n-1}, \langle \aleph_0 \rangle) \leadsto P \), holds. In particular, \( \text{kur}_0(P) \leq \dim(P) \).

In particular, calculations of critical points may lead to estimates of the form \( \text{crit}_r(\mathcal{A}; \mathcal{B}) \leq \aleph_{\log \log n} \ldots \)
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**Theorem (Jónsson, 1962)**

Let $L$ be a sectionally complemented modular lattice with a large 4-frame. If $L$ has a countable cofinal sequence, then $L$ is coordinatizable (i.e., $\exists$ regular ring such that $L \cong L(R)$).

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Larders and CLL

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Then larders are used to turn the diagram counterexample to an object counterexample.
Description of the larder data

A modification of the following (with $\lambda := \aleph_1$):
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$$
\begin{align*}
S &\Rightarrow := SCML \Rightarrow \\
S &:= SCML \\
A &:= SCML \\
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B &:= Reg \\
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\Phi &:= id \\
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\end{align*}
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An extension $A \leq B$ of algebras is **congruence-preserving**, if the canonical map $\text{Con } A \rightarrow \text{Con } B$ is an isomorphism.
Lattices without congruence-permutable, congruence-preserving extension

An extension \( A \leq B \) of algebras is *congruence-preserving*, if the canonical map \( \text{Con} A \to \text{Con} B \) is an isomorphism.

**Theorem (Gillibert and W., 2009)**

Due to earlier results of Ploščica, Tůma, and W., the analogue of this result at \( \aleph_2 \) was already known. Furthermore, in case \( V \) is locally finite, then \( \aleph_1 \) is optimal in the result above. (Open problem in the non locally finite case. For example: does the free lattice on \( \aleph_0 \) generators have a congruence-permutable, congruence-preserving extension?). Unlike all previous examples, the larder data are difficult to figure out.

Let's give an outline.
An extension $A \leq B$ of algebras is **congruence-preserving**, if the canonical map $\text{Con} A \to \text{Con} B$ is an isomorphism.

**Theorem (Gillibert and W., 2009)**

Let $\mathcal{V}$ be a nondistributive lattice variety. Then the free lattice (resp., the free bounded lattice) on $\aleph_1$ generators within $\mathcal{V}$ has no congruence-permutable, congruence-preserving extension.
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Due to earlier results of Ploščica, Tůma, and W., the analogue of this result at $\aleph_2$ was already known. Furthermore, in case $\mathcal{V}$ is locally finite, then $\aleph_1$ is optimal in the result above. (Open problem in the non locally finite case. For example: does the free lattice on $\aleph_0$ generators have a congruence-permutable, congruence-preserving extension?)
Lattices without congruence-permutable, congruence-preserving extension

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Unlike all previous examples, the larder data are difficult to figure out. Let’s give an outline.
A semilattice-metric space is a triple $A = (A, \delta_A, \tilde{A})$, where $A$ is a set, $\tilde{A}$ is a $(\lor, 0)$-semilattice, $\delta_A : A \times A \to \tilde{A}$, $\delta_A(x, x) = 0$, $\delta_A(x, y) = \delta_A(y, x)$, $\delta_A(x, z) \leq \delta_A(x, y) \lor \delta_A(y, z)$ $\forall x, y, z \in A$ (say that $\delta_A$ is a distance).
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Morphisms: $(f, \tilde{f}) : \mathbf{A} \to \mathbf{B}$ means that $f : A \to B$, $\tilde{f} : \tilde{A} \to \tilde{B}$, and $\delta_B(f(x), f(y)) = \tilde{f} \delta_A(x, y)$ $\forall x, y \in A$. 
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Semilattice-metric spaces

- A **semilattice-metric space** is a triple $\mathbf{A} = (A, \delta_{\mathbf{A}}, \tilde{A})$, where $A$ is a set, $\tilde{A}$ is a $(\lor, 0)$-semilattice, $\delta_{\mathbf{A}} : A \times A \to \tilde{A}$, $\delta_{\mathbf{A}}(x, x) = 0$, $\delta_{\mathbf{A}}(x, y) = \delta_{\mathbf{A}}(y, x)$, $\delta_{\mathbf{A}}(x, z) \leq \delta_{\mathbf{A}}(x, y) \lor \delta_{\mathbf{A}}(y, z) \ \forall x, y, z \in A$ (say that $\delta_{\mathbf{A}}$ is a distance).

- **Morphisms**: $(f, \tilde{f}) : \mathbf{A} \to \mathbf{B}$ means that $f : A \to B$, $\tilde{f} : \tilde{A} \to B$, and $\delta_{\mathbf{B}}(f(x), f(y)) = \tilde{f} \delta_{\mathbf{A}}(x, y) \ \forall x, y \in A$. We get a category, **Metr**.

- Double arrows in **Metr**: $(f, \tilde{f}) : \mathbf{A} \to \mathbf{B}$ such that $f$ is surjective (nothing said about $\tilde{f}$).
A semilattice-metric cover is a quadruple $\mathbf{A} = (A^*, A, \delta_\mathbf{A}, \tilde{A})$, where $A^* \subseteq A$, $(A, \delta_\mathbf{A}, \tilde{A})$ is a semilattice-metric space, and $\forall x, y, z \in A^*$, $\exists t \in A$ such that $\delta_\mathbf{A}(x, t) \leq \delta_\mathbf{A}(y, z)$ and $\delta_\mathbf{A}(t, z) \leq \delta_\mathbf{A}(x, y)$ (Parallelogram Rule: imitates one step of “congruence-permutable”).
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Morphisms defined as in $\text{Metr}$, with $f(A^*) \subseteq B^*$. Get a category $\text{Metr}^*$. 
Semilattice-metric spaces and covers

- A **semilattice-metric cover** is a quadruple $\mathcal{A} = (A^*, A, \delta_A, \tilde{A})$, where $A^* \subseteq A$, $(A, \delta_A, \tilde{A})$ is a semilattice-metric space, and $\forall x, y, z \in A^*$, $\exists t \in A$ such that $\delta_A(x, t) \leq \delta_A(y, z)$ and $\delta_A(t, z) \leq \delta_A(x, y)$ (Parallelogram Rule: imitates one step of “congruence-permutable”).

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- “Forgetful” functor $\psi : \textbf{Metr}^* \rightarrow \textbf{Metr}$, $A \mapsto (A^*, \delta_A|_{A^* \times A^*}, \tilde{A})$. 

- Ladders and CLL

- General settings

- $P$-scaled algebras

- Lifters, ladders, and CLL

- Diagram form of GS

- Relative critical points

- Non-coordinatizable SCMLs

- Lattices without CPCP-extension
Every algebra $A$ defines canonically a semilattice-metric space $\Phi(A) := (A, \text{con}_A, \text{Con}_c A)$, where $\text{con}_A(x, y)$ denotes the (principal) congruence generated by $(x, y)$. 

From algebras to semilattice-metric spaces
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For algebras $A$ and $B$ with $\text{Op}(A) \subseteq \text{Op}(B)$, a morphism $f : A \rightarrow B$ is a map $A \rightarrow B$ which is a homomorphism for each symbol in $\text{Op}(A)$. This way we get a category, $\mathbf{MAlg}$. 
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For algebras $A$ and $B$ with $\text{Op}(A) \subseteq \text{Op}(B)$, a morphism $f : A \to B$ is a map $A \to B$ which is a homomorphism for each symbol in $\text{Op}(A)$. This way we get a category, $\text{MAlg}$. Then $\Phi$ extends naturally to a functor $\text{MAlg} \to \text{Metr}$. 
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Picture of the larder data

\[ S \Rightarrow := \text{Metr} \Rightarrow \]
\[ S := \text{Metr} \]

\[ \Phi \]

\[ \Psi \]

\[ \mathcal{A} := \text{MAlg} \]
\[ \mathcal{A}^\dagger := \text{MAlg}_{\text{fin}} \]

\[ \mathcal{B} := \text{Metr}^* \]
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Hard core of the proof 1: a square of finite lattices

The lattices in the two following diagrams have no CPCP-extension that would be functorial wrt. those diagrams:
Hard core of the proof 2: another square of finite lattices

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