

Recent progress on congruence lattices of infinite lattices

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Main reference:

J. Tůma and F. Wehrung

“A survey of recent results on congruence
lattices of lattices”

Algebra Universalis **48** (2002), 439–471.

For a lattice L , the set $\text{Con } L$ of all congruences of L , partially ordered by inclusion, is an algebraic lattice. Its compact elements are the *finitely generated congruences*, that is, the congruences of the form

$$\bigvee_{i < n} \Theta_L(a_i, b_i),$$

where $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in L$ and $\Theta_L(a_i, b_i)$ denotes the principal congruence of L generated by $\langle a_i, b_i \rangle$.

We denote by $\text{Con}_c L$ the $\langle \vee, 0 \rangle$ -semilattice of all compact congruences of L .

$\text{Con } L \cong \text{Id}(\text{Con}_c L)$, the ideal lattice of $\text{Con}_c L$.

Problems about $\text{Con } L$ are often more conveniently stated in terms of $\text{Con}_c L$.

N. Funayama and T. Nakayama proved, in 1942, the following.

Proposition 1 *The lattice $\text{Con } L$ is distributive, for every lattice L .*

Nontrivial direction: prove the containment

$$a \cap (b \vee c) \subseteq (a \cap b) \vee (a \cap c),$$

for all $a, b, c \in \text{Con } L$.

Use the following lattice polynomial:

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

It is a *majority operation* on any lattice:

$$m(y, x, x) = m(x, y, x) = m(x, x, y) = x.$$

If $x \equiv_{a \wedge (b \vee c)} y$, then there exists a finite sequence

$$x = z_0 \equiv_b z_1 \equiv_c \cdots \equiv_c z_{2n} = y.$$

Put $t_i = m(x, y, z_i)$. Then

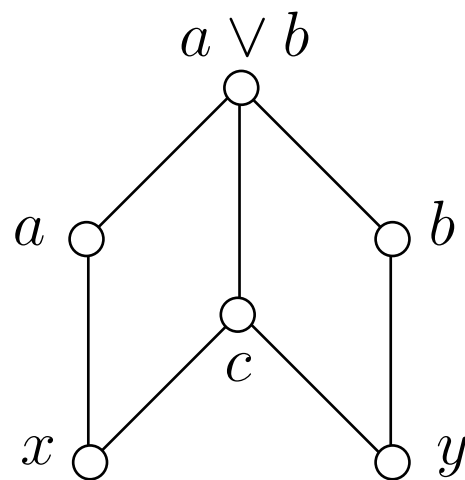
$$x = t_0 \equiv_{a \wedge b} t_1 \equiv_{a \wedge c} \cdots \equiv_{a \wedge c} t_{2n} = y,$$

so $x \equiv_{(a \cap b) \vee (a \cap c)} y$.

Semilattice-theoretical translation:

Proposition 2 *The semilattice $\text{Con}_c L$ is distributive, for every lattice L .*

A $\langle \vee, 0 \rangle$ -semilattice S is *distributive*, if for all $a, b, c \in S$, if $c \leq a \vee b$, then there are $x \leq a$ and $y \leq b$ such that $c = x \vee y$.



Equivalently, the ideal lattice $\text{Id } S$ of S is distributive.

The Congruence Lattice Problem (CLP) was formulated by R. P. Dilworth in 1945.

Problem 1 (CLP) *Is every algebraic distributive lattice isomorphic to $\text{Con } L$, for some lattice L ?*

Semilattice-theoretical formulation:

Problem 1' *Is every distributive $\langle \vee, 0 \rangle$ -semilattice isomorphic to $\text{Con}_c L$, for some lattice L ?*

For finite L , congruences of L can be easily computed in terms of the *join-dependency relation* D on L . For $a \neq b$ in $J(L)$, $a D b$ if there exists $x \in L$ such that $a \leq b \vee x$ is *minimal* in b . Let \trianglelefteq denote the reflexive, transitive closure of D . Then, for $p, q \in J(L)$,

$$\Theta_L(p_*, p) \subseteq \Theta_L(q_*, q) \iff p \trianglelefteq q.$$

Now let S be a finite, distributive $\langle \vee, 0 \rangle$ -(semi)-lattice. A subset $X \subseteq J(S) \times 3$ is *closed*, if for all distinct $i, j, k < 3$ and all $p, q, r \in J(S)$ with $p \leq q, r$ and $p \in \{q, r\}$,

$$\langle q, j \rangle, \langle r, k \rangle \in X \implies \langle p, i \rangle \in X.$$

Let $\Phi(S)$ denote the lattice of all closed subsets of $J(S) \times 3$.

Proposition 3 *The lattice $\Phi(S)$ is finite atomistic, and $\text{Con } \Phi(S) \cong S$.*

G. Grätzer and E. T. Schmidt (1962) obtained, for every finite S , a finite, *sectionally complemented* L with $\text{Con } L \cong S$ and $|J(L)| < 2|J(S)|$. The best possible constant for L finite is given by $|J(L)| \leq (5/3)|J(S)|$ (G. Grätzer and F. Wehrung, 1999).

Now we move to infinite lattices.

Lemma 4 (P. Pudlák, 1985) *Every distributive $\langle \vee, 0 \rangle$ -semilattice is the directed union of its finite distributive $\langle \vee, 0 \rangle$ -subsemilattices.*

This suggests to combine Proposition 3 and Lemma 4:

- (1) Express an arbitrary distributive $\langle \vee, 0 \rangle$ -semilattice S as the directed union of its finite distributive $\langle \vee, 0 \rangle$ -subsemilattices.
- (2) Define $\Phi(S)$ as $\varinjlim \langle \Phi(X) \mid X \subseteq S \text{ finite} \rangle$.
- (3) Hope that $\text{Con}_c \Phi(S) \cong S$.

Item (1) is Pudlák's Lemma.

Item (3): for every lattice homomorphism $f: K \rightarrow L$, let $\text{Con}_c f: \text{Con}_c K \rightarrow \text{Con}_c L$ be the $\langle \vee, 0 \rangle$ -homomorphism defined by the rule

$$(\text{Con}_c f)(a) = \bigvee_{\langle x, y \rangle \in a} \Theta_L(f(x), f(y)).$$

Lemma 5 *The correspondence $K \mapsto \text{Con}_c K$, $f \mapsto \text{Con}_c f$ is a functor, which preserves direct limits. In particular, $\text{Con}_c \varinjlim_{i \in I} L_i \cong \varinjlim_{i \in I} \text{Con}_c L_i$.*

And Item (2)?

Reminder: for finite S , $\Phi(S)$ is defined as the lattice of all “closed” subsets of $J(S) \times 3$. This looks like a quite “natural” definition.

However, Φ is not a functor (from semilattices to lattices). Thus there is no way to define anything like $\varinjlim \langle \Phi(X) \mid X \subseteq S \text{ finite} \rangle$!

Problem: define $\Phi(f)$, where $f: A \rightarrow B$ is a $\langle \vee, 0 \rangle$ -semilattice-embedding.

In case f is a *lattice* embedding, the dual function $f^*: J(B) \rightarrow J(A)$ defined by

$$f^*(q) = \text{least } p \in A \text{ such that } q \leq f(p)$$

is order-preserving, defined on a hereditary subset of $J(B)$, and surjective. This is still not sufficient to define a nice lattice homomorphism $\Phi(A) \rightarrow \Phi(B)$. For this, we need f^* to be *proper*:

$$f^*(x) \leq f^*(y) \Rightarrow (\exists z)(x \leq z \text{ and } f^*(z) = f^*(y)).$$

By lifting order-preserving surjective functions (between finite posets) by *proper* homomorphisms of finite reflexive relational systems, P. Pudlák obtains the following result.

Theorem 6 *There exists a functor Ψ , from distributive 0-lattices and $\langle \vee, \wedge, 0 \rangle$ -embeddings, to 0-lattices and $\langle \vee, \wedge, 0 \rangle$ -embeddings with CEP, such that $\text{Con}_c \circ \Psi$ is naturally equivalent to the identity.*

Illustration of the natural equivalence:

$$\begin{array}{ccc}
 A & \xrightarrow[\vee, \wedge, 0]{f} & B \\
 \downarrow \cong & & \downarrow \cong \\
 \text{Con}_c \Psi(A) & \xrightarrow{\text{Con}_c \Psi(f)} & \text{Con}_c \Psi(B)
 \end{array}$$

In particular, $\text{Con}_c \Psi(S) \cong S$, for every distributive *lattice* S with zero. Furthermore, $\Psi(S)$ is a direct limit of finite, atomistic lattices and lattice embeddings with CEP.

Corollary 7 (E. T. Schmidt, 1981) *Every distributive 0-lattice S is isomorphic to $\text{Con}_c L$, for some lattice L .*

The lattice $L = \Psi(S)$ obtained by Pudlák's method has some additional properties: $L = \bigcup_{i \in I} L_i$ (directed union), where the L_i -s are finite atomistic. In particular, L is *locally finite*.

Schmidt's method proceeds differently. It establishes that $S \cong B/\theta$, for some generalized Boolean semilattice B and some *distributive* \vee -congruence θ of B . Schmidt's Lemma says that every such semilattice is representable (as $\text{Con}_c L$ for some lattice L).

However, the methods above are not sufficient to solve CLP.

Theorem 8 (F. Wehrung, 1999) *There exists a distributive $\langle \vee, 0 \rangle$ -semilattice S , of size \aleph_2 , with the following properties:*

- (i) *$S \not\cong \text{Con}_c L$, for any direct limit L of finite atomistic lattices, or any sectionally complemented lattice, or any relatively complemented lattice.*
- (ii) *S is not a (weakly) distributive image of any generalized Boolean semilattice.*

In fact, one can take $S = \text{Con}_c F_{\mathbf{V}}(\omega_2)$, where \mathbf{V} is any non-distributive variety of lattices (M. Ploščica, J. Tůma, and F. Wehrung, 1998). The negative property used is a *uniform refinement property*, WURP, not satisfied by that S .

Let a distributive $\langle \vee, 0 \rangle$ -semilattice S satisfy WURP, if for all $e \in S$ and all families $\langle a_i \mid i \in I \rangle$ and $\langle b_i \mid i \in I \rangle$ such that $a_i \vee b_i = e$ for all $i \in I$, there exists a family $\langle c_{i,j} \mid i \in I \rangle$ of elements of S such that for all $i, j, k \in I$,

$$(i) \quad c_{i,j} \leq a_i, b_j,$$

$$(ii) \quad a_j \vee b_i \vee c_{i,j} = e,$$

$$(iii) \quad c_{i,k} \leq c_{i,j} \vee c_{j,k}.$$

The following is a consequence of work from M. Ploščica, J. Tůma, and F. Wehrung (1998) and J. Tůma and F. Wehrung (2001).

Theorem 9

- (i) *For every lattice L with permutable congruences, $\text{Con}_c L$ satisfies WURP.*
- (ii) *For every non-distributive variety \mathbb{V} of lattices, $\text{Con}_c F_{\mathbb{V}}(\omega_2)$ does not satisfy WURP.*

For example, every direct limit of finite atomistic lattices, or every sectionally complemented lattice, or every relatively complemented lattice, has permutable congruences.

Furthermore, the size \aleph_2 is optimal in the result above.

Many results on congruence classes of finitely generated varieties are due to M. Ploščica. For $n \geq 3$, denote by $M_n^{0,1}$ the variety of *bounded* lattices generated by the lattice M_n (consisting of 0, 1, and n atoms).

Theorem 10 *For every $K \in M_n^{0,1}$ of cardinality \aleph_1 , there exists $L \in M_3^{0,1}$ such that $\text{Con}K \cong \text{Con}L$. The analogue for \aleph_2 fails.*

For a class \mathbf{U} of lattices, denote by $\text{Con}_c \mathbf{U}$ the class of all semilattices isomorphic to some $\text{Con}_c S$, for $S \in \mathbf{U}$.

Problem 2 (Critical Point Conjecture) *Let \mathbf{U} and \mathbf{V} be (finitely generated) lattice varieties. If $\text{Con}_c \mathbf{U}$ and $\text{Con}_c \mathbf{V}$ agree on all countable semilattices, do they agree on all semilattices of cardinality \aleph_1 ?*

This conjecture holds for all known examples, even for finitely generated varieties.

The following is a combination of results from E. T. Schmidt, P. A. Grillet, E. G. Effros, D. E. Handelman, C.-L. Shen, G. M. Bergman.

Theorem 11 *Every countable $\langle \vee, 0 \rangle$ -semilattice is isomorphic to $\text{Con}_c L$, for some locally finite, sectionally complemented, modular lattice L .*

The proof illustrates the use of category theory on the Con_c functor.

The following has been established by various authors, including S. Bulman-Fleming (1978, methods of category theory), and K. R. Goodearl and F. Wehrung (2001, methods of universal algebra).

Lemma 12 *Every distributive $\langle \vee, 0 \rangle$ -semilattice is a direct limit of finite Boolean semilattices and $\langle \vee, 0 \rangle$ -homomorphisms.*

The central idea of the proof of Lemma 12 is to establish that for every distributive $\langle \vee, 0 \rangle$ -homomorphism S and every $m < \omega$, every $\langle \vee, 0 \rangle$ -homomorphism $f: \mathbf{2}^m \rightarrow S$, there are $n < \omega$ and $\langle \vee, 0 \rangle$ -homomorphisms $e: \mathbf{2}^m \rightarrow \mathbf{2}^n$ and $g: \mathbf{2}^n \rightarrow S$ such that $f = g \circ e$ and $\ker e = \ker f$.

$$\begin{array}{ccc}
 \mathbf{2}^m & \xrightarrow{f} & S \\
 \downarrow e & \nearrow g & \\
 \mathbf{2}^n & &
 \end{array}$$

To prove this, it suffices to establish that finite systems of equations of the form

$$\bigvee_{i \in I} x_i = \bigvee_{j \in J} x_j$$

can be “parametrized” in all distributive semilattices (find “most general solutions”).

Example: $x \vee z = y \vee z$ can be parametrized in all distributive semilattices, by

$$x = t \vee u; \quad y = t \vee v; \quad z = u \vee v \vee w.$$

In order to prove Theorem 11, we express the original semilattice S as a direct limit $S = \varinjlim_{n < \omega} S_n$, where the S_n -s are finite Boolean.

The S_n -s are easy to lift *individually*. For example, $\text{Con } 2^n \cong 2^n$. However, they need to be lifted *collectively*—that is, together with the transition maps $S_n \rightarrow S_{n+1}$.

Definition 13 *Let \mathbb{F} be a field. A \mathbb{F} -lattice is a lattice of the form*

$$\prod_{i < n} \text{Sub } V_i,$$

where the V_i -s are nontrivial finite-dimensional \mathbb{F} -vector spaces. (Sub V is the subspace lattice of V .)

For a lattice L , put $|\theta| = 1$ if θ is nonzero, 0 otherwise, for all $\theta \in \text{Con } L$. For a \mathbb{F} -lattice $L = \prod_{i < n} L_i$, with L_i simple, we have $\text{Con } L \cong 2^n$, via the map

$$\prod_{i < n} \theta_i \mapsto \langle |\theta_i| \mid i < n \rangle.$$

Now a *one-dimensional amalgamation result*.

Lemma 14 *Let \mathbb{F} be a field, let K be a \mathbb{F} -lattice, let $n < \omega$, and let $f: \text{Con } K \rightarrow 2^n$ be a $\langle \vee, 0 \rangle$ -homomorphism. Then f can be lifted, that is, there are a \mathbb{F} -lattice L , a 0-lattice homomorphism $f: K \rightarrow L$, and an isomorphism $\varepsilon: \text{Con } L \rightarrow 2^n$ such that $f = \varepsilon \circ \text{Con } f$.*

Given:

$$\text{Con } K \xrightarrow{f} 2^n$$

Obtained:

$$\begin{array}{c} L \\ \uparrow f \\ K \end{array}$$

$$\begin{array}{ccc} & \text{Con } L & \\ & \uparrow & \searrow \varepsilon \cong \\ \text{Con } f & & \\ \uparrow & & \\ \text{Con } K & \xrightarrow{f} & 2^n \end{array}$$

To prove Lemma 14, we start with the case where $n = 1$. Write $K = \prod_{i < m} K_i$, with $K_i = \text{Sub } V_i$. So f is given by the rule

$$f\left(\prod_{i < m} \theta_i\right) = \bigvee_{i \in I} |\theta_i|,$$

for some $I \subseteq m$.

Put $V = \bigoplus_{i < m} V_i$ and $L = \text{Sub } V$. Define $f: K \rightarrow L$:

$$f(\langle X_i \mid i < m \rangle) = \bigoplus_{i \in I} X_i \quad (\text{all } X_i \in \text{Sub } V_i).$$

Since L is simple, we take $\varepsilon: \text{Con } L \rightarrow \mathbf{2}$, $\theta \mapsto |\theta|$. Then $f = \varepsilon \circ \text{Con } f$.

To finish the proof of Lemma 14: apply the previous slide to the n components $f_i: \text{Con } K \rightarrow \mathbf{2}$ of f . Get simple \mathbb{F} -lattices L_i , 0-lattice homomorphisms $f_i: K \rightarrow L_i$, and isomorphisms $\varepsilon_i: \text{Con } L_i \rightarrow \mathbf{2}$, $\theta \mapsto |\theta|$.

$$\begin{array}{ccc}
 & L_i & \text{Con } L_i \\
 & \uparrow f_i & \uparrow \text{Con } f_i \\
 \text{Con } K \xrightarrow{f_i} \mathbf{2} & & \text{Con } K \xrightarrow{f_i} \mathbf{2} \\
 & K & \searrow \varepsilon_i
 \end{array}$$

Put $L = \prod_{i < n} L_i$, $f(x) = \langle f_i(x) \mid i < n \rangle$, and $\varepsilon(\prod_{i < n} \theta_i) = \langle |\theta_i| \mid i < n \rangle$.

$$\begin{array}{ccc}
 & L & \text{Con } L \\
 & \uparrow f & \uparrow \text{Con } f \\
 \text{Con } K \xrightarrow{f} \mathbf{2}^n & & \text{Con } K \xrightarrow{f} \mathbf{2}^n \\
 & K & \searrow \varepsilon
 \end{array}$$

Now finishing the proof of Theorem 11! Start with a countable distributive $\langle \vee, 0 \rangle$ -semilattice S . By Lemma 12, $S = \varinjlim_{n < \omega} S_n$, with the S_n -s finite Boolean, transition maps $e_n: S_n \rightarrow S_{n+1}$, and limiting maps $f_n: S_n \rightarrow S$.

Pick any \mathbb{F} -lattice L_0 with an isomorphism $\varepsilon_0: \text{Con } L_0 \rightarrow S_0$. Apply Lemma 14 to $f_0 \circ \varepsilon_0: \text{Con } L_0 \rightarrow S_1$. We obtain a \mathbb{F} -lattice L_1 , a 0-lattice homomorphism $f_0: L_0 \rightarrow L_1$, and an isomorphism $\varepsilon_1: \text{Con } L_1 \rightarrow S_1$ such that $\varepsilon_1 \circ \text{Con } f_0 = f_0 \circ \varepsilon_0$. Proceed with L_1 instead of L_0 . We obtain a commutative diagram:

$$\begin{array}{ccccc}
 \text{Con } L_0 & \xrightarrow{\text{Con } f_0} & \text{Con } L_1 & \xrightarrow{\text{Con } f_1} & \dots \\
 \varepsilon_0 \downarrow & & \varepsilon_1 \downarrow & & \\
 S_0 & \xrightarrow{f_0} & S_1 & \xrightarrow{f_1} &
 \end{array}$$

Then $L = \varinjlim_{n < \omega} L_n$ (with transition maps $f_n: L_n \rightarrow L_{n+1}$) satisfies $\text{Con}_c L \cong S$.

Important information used in the process:

The $(L \mapsto \text{Con}_c L, f \mapsto \text{Con}_c f)$ functor preserves direct limits.

Additional information on the solution lattice L :

It is a $\langle \vee, \wedge, 0 \rangle$ -direct limit of \mathbb{F} -lattices. In particular, it is sectionally complemented (i.e., $(\forall a \leq b)(\exists x)(a \oplus x = b)$) and modular.

In addition, *if \mathbb{F} is finite, then L is locally finite.*

This has been extended to distributive 0-lattices by P. Růžička in 2000.

Theorem 15 *Let \mathbb{F} be a field. Then every distributive lattice with zero is isomorphic to $\text{Con}_c L$, for some direct limit L of \mathbb{F} -lattices.*

Unlike the proof of Pudlák's Theorem (Theorem 6), Růžička's construction is not functorial.

Corollary 16 *Every distributive lattice with zero is isomorphic to $\text{Con}_c L$, for some locally finite, sectionally complemented, modular lattice L .*

This leads to the following question:

Question: is every distributive $\langle \vee, 0 \rangle$ -semilattice isomorphic to $\text{Con}_c L$, for some *locally finite modular* lattice L ?

(Sectionally complemented lattices are ruled out, by Theorem 8.)

Answer: NO, with a counterexample of size \aleph_2 due to P. Růžička (2000).

Size brought down to \aleph_1 by F. Wehrung in 2002.

Description of the latter counterexample:

Let \mathcal{B} denote the Boolean algebra generated by all intervals of ω_1 . Let \mathcal{J} (resp., \mathcal{F}) consist of all bounded (resp., unbounded) members of \mathcal{B} . Put

$$D = \{x \subseteq \omega_1 \mid \text{either } x \text{ is finite or } x = \omega_1\}.$$

The counterexample is

$$S_{\omega_1} = (\{\emptyset\} \times \mathcal{J}) \cup ((D \setminus \{\emptyset\}) \times \mathcal{F}).$$

So, elements of S_{ω_1} are of one of the two forms

$$\begin{aligned} &\langle \emptyset, \text{bounded} \rangle, \\ &\langle \neq \emptyset, \text{unbounded} \rangle. \end{aligned}$$

How does this counterexample work?

Any lattice L satisfies $\text{Con}_c L \cong \text{Dim } L / \asymp$, where $\text{Dim } L$ is the *dimension monoid* of L and

$$x \asymp y \Leftrightarrow (\exists n)(x \leq ny \text{ and } y \leq nx).$$

In case L is locally finite and modular, $\text{Dim } L$ is the positive cone of a *dimension group*.

And for any dimension group G , the maximal semilattice quotient $S = G^+ / \asymp$ satisfies the following infinitary statement, another “uniform refinement property”, not satisfied by S_{ω_1} .

For all $e \in S$ and all subsets A and B of S such that A is uncountable, B is \aleph_0 -downward directed, and

$$a \leq e \leq a \vee b \text{ for all } \langle a, b \rangle \in A \times B,$$

there exists $a \in A^{[\wedge 2]}$ such that $e \leq a \vee b$ for all $b \in B$.

Here, $A^{[\wedge 2]}$ denotes the set of finite joins of elements $x \leq u, v$, with $u \neq v$ in A .

How to represent semilattices of size \aleph_1 ?

The main tool, due to H. Dobbertin and A. P. Huhn, is the notion of “ladder” (first called “frame”).

Definition 17 *Let $m \in \omega \setminus \{0\}$. A m -ladder is a lattice F with zero with the following properties:*

- (i) *$[0, a]$ is finite, for all $a \in F$.*
- (ii) *Every element of F has at most m immediate predecessors.*

Example. The 1-ladders are exactly the initial segments of ω . They are all (at most) countable.

Kuratowski's free set Lemma dates back to 1951. It is instrumental in the proofs of Theorems 8 and 9.

Lemma 18 *Let $n \in \omega \setminus \{0\}$, let $|X| \geq \aleph_n$, and let $f: [X]^n \rightarrow [X]^{<\omega}$. There exists $U \in [X]^{n+1}$ such that $u \notin f(U \setminus \{x\})$ for all $x \in U$.*

And conversely.

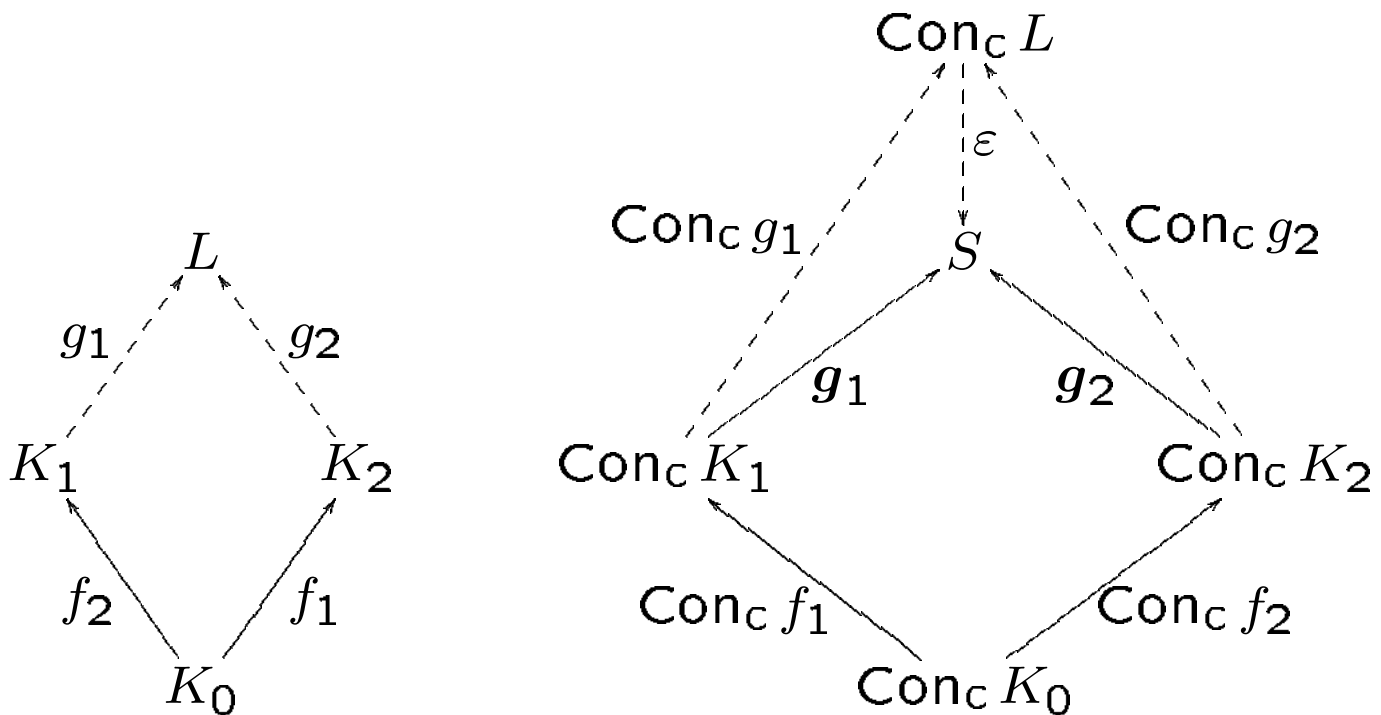
Taking $f(Y) = \bigvee Y$, we obtain the following.

Corollary 19 *Let $m \in \omega \setminus \{0\}$. Every m -ladder has cardinality at most \aleph_{m-1} .*

It is easy to construct a 2-ladder of cardinality \aleph_1 .

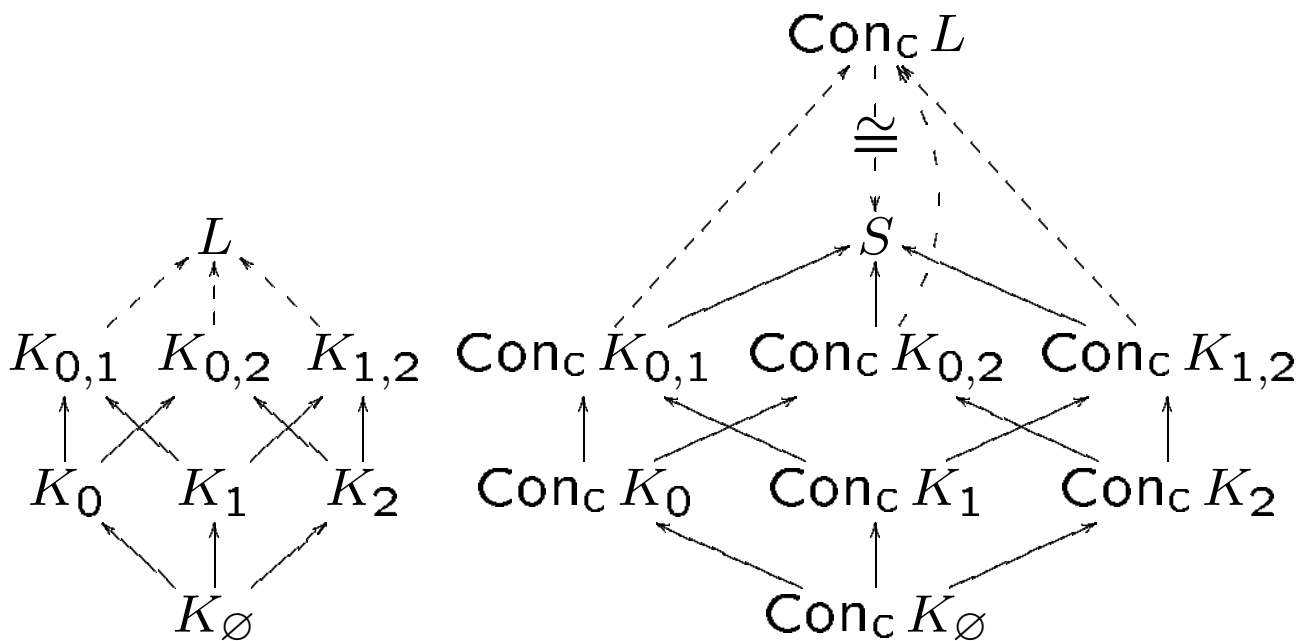
Romantic open problem (S. Z. Ditor, 1984). Does there exist a 3-ladder of cardinality \aleph_2 ?

Say that a $\langle \vee, 0 \rangle$ -semilattice S satisfies 2-CLP (2-dimensional amalgamation property), if for all lattices K_0, K_1, K_2 , all lattice homomorphisms $f_i: K_0 \rightarrow K_i$ and all $\langle \vee, 0 \rangle$ -homomorphisms $g_i: \text{Con}_c K_i \rightarrow S$ ($i \in \{1, 2\}$) such that $g_1 \circ \text{Con}_c f_1 = g_2 \circ \text{Con}_c f_2$, there are a lattice L , lattice homomorphisms $g_i: K_i \rightarrow L$, and an isomorphism $\varepsilon: \text{Con}_c L \rightarrow S$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $\varepsilon \circ \text{Con}_c g_i = g_i$ for all $i \in \{1, 2\}$.



And 3-CLP?

Formulated with truncated cubes instead of truncated squares.



Answer: only the one-element semilattice has 3-CLP.

The following is due to J. Tůma (1998) and G. Grätzer, H. Lakser, and F. Wehrung (2000).

Theorem 20 *Every finite distributive $\langle \vee, 0 \rangle$ -semilattice has the 2-CLP. This also works for finite lattices.*

By using this, one can extend the proof of the representation for $|S| \leq \aleph_0$ to the case where $|S| \leq \aleph_1$. Instead of using the index set ω and 1-dimensional amalgamation, we use a 2-ladder of cardinality \aleph_1 and 2-dimensional amalgamation. We obtain the following, first proved by A. P. Huhn in 1989 without restriction on the lattice.

Corollary 21 *Every distributive $\langle \vee, 0 \rangle$ -semilattice of cardinality $\leq \aleph_1$ is isomorphic to $\text{Con}_c L$, for some locally finite, relatively complemented lattice L with zero, bounded if S is bounded.*

By previous results, one cannot strengthen “locally finite” to “locally finite and modular”.

However, there is an analogue of Theorem 20 for von Neumann regular rings (the hard part of the work was done by P. M. Cohn in 1959). By using again a 2-ladder and 2-dimensional amalgamation, we obtain the following.

Theorem 22 *Every distributive $\langle \vee, 0 \rangle$ -semilattice of cardinality $\leq \aleph_1$ is isomorphic to $\text{Con}_c L$, for some relatively complemented modular lattice L with zero, bounded if S is bounded.*

The L constructed there is not locally finite. It has the form

$$L(R) = \{xR \mid x \in R\},$$

for some von Neumann regular ring R .

**Now let's move beyond distributive lattices
and cardinality $\leq \aleph_1 \dots$**

Definition 23 A partial prelattice is a structure $\langle P, \leq, \vee, \wedge \rangle$ such that

- (i) \leq is a quasi-ordering on P .
- (ii) Both \vee and \wedge are partial functions from $[P]_*^{<\omega} = [P]^{<\omega} \setminus \{\emptyset\}$ to P .
- (iii) $a = \vee X$ (resp., $a = \wedge X$) implies that a is “the” least upper bound (resp., greatest lower bound) of X with respect to \leq .

We say that P is a partial lattice, if \leq is antisymmetric. A congruence of P is a quasi-ordering \preceq on P containing \leq such that $\langle P, \preceq, \vee, \wedge \rangle$ is a partial prelattice.

For $a, b \in P$, we denote by $\Theta^+(a, b)$ the least congruence θ of P such that $a \leq_\theta b$. In lattices,

$$\Theta^+(a, b) = \Theta(a \wedge b, a) = \Theta(b, a \vee b).$$

It is easy to define a functor $P \mapsto \text{Con}_c P$, $f \mapsto \text{Con}_c f$ from partial lattices to $\langle \vee, 0 \rangle$ -semi-lattices. The elements of $\text{Con}_c P$ are finite joins of the form

$$\bigvee_{i < n} \Theta^+(a_i, b_i), \text{ with all } a_i, b_i \in P.$$

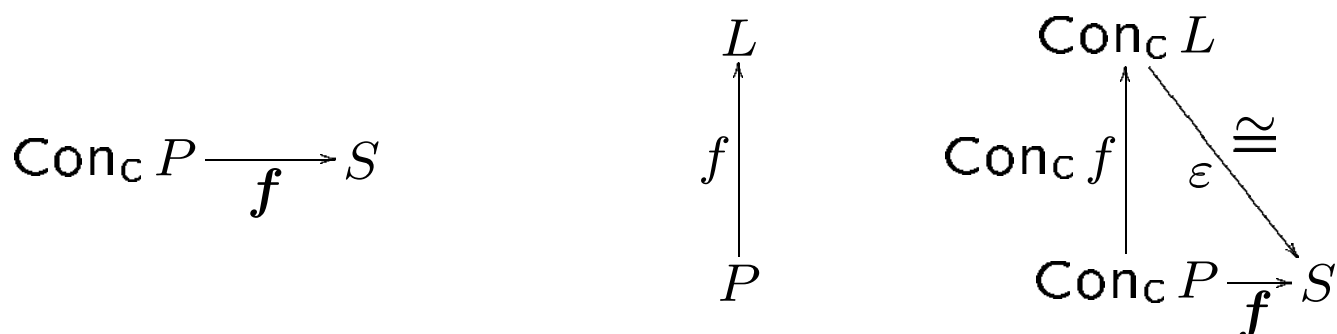
Note that $\text{Con}_c P$ may not be distributive.

Proposition 24 *The Con_c functor on partial lattices preserves direct limits.*

Now what are we trying to do?

Typical problem. Try to *lift* a given $\langle \vee, 0 \rangle$ -homomorphism $f: \text{Con}_c P \rightarrow S$. That is, find a lattice L , a homomorphism $f: P \rightarrow L$ of partial lattices, and an isomorphism $\varepsilon: \text{Con}_c L \rightarrow S$ such that $\varepsilon \circ \text{Con}_c f = f$.

Given: Obtained:



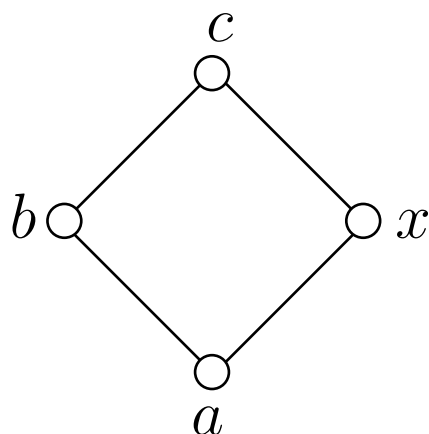
Definition 25 We say that a $\langle \vee, 0 \rangle$ -semilattice S has 1-CLP, if this can always be done whenever P is a (total) lattice.

The proofs proceed by enlarging P step by step “to make f an isomorphism”, occasionally replacing P by the free lattice $F_L(P)$ over P .

Replacing P by $F_L(P)$. The canonical map $j_P: P \hookrightarrow F_L(P)$ has the CEP. If S is an *injective* $\langle \vee, 0 \rangle$ -semilattice, then there exists $g: \text{Con}_c F_L(P) \rightarrow S$ such that $g \circ \text{Con}_c j_P = f$.

$$\begin{array}{ccc}
 \text{Con}_c P & \xrightarrow{f} & S \\
 \text{Con}_c j_P \downarrow & \nearrow g & \\
 \text{Con}_c F_L(P) & &
 \end{array}$$

Adding a relative complement. We are given $a < b < c$ in P , we find an extension of P with a relative complement of b in $[a, c]$ that does not destroy the congruence lattice of P .



We do this *freely*, that is, put $Q = P \cup \{x\}$, where x is free such that $b \vee x = c$ and $b \wedge x = a$. Let $e_{P,Q}: \text{Con}_c P \rightarrow \text{Con}_c Q$ denote the canonical map.

Again, if S is an injective semilattice, then there exists $g: \text{Con}_c Q \rightarrow S$ such that $g \circ e_{P,Q} = f$.

$$\begin{array}{ccc}
 \text{Con}_c P & \xrightarrow{f} & S \\
 e_{P,Q} \downarrow & \nearrow g & \\
 \text{Con}_c Q & &
 \end{array}$$

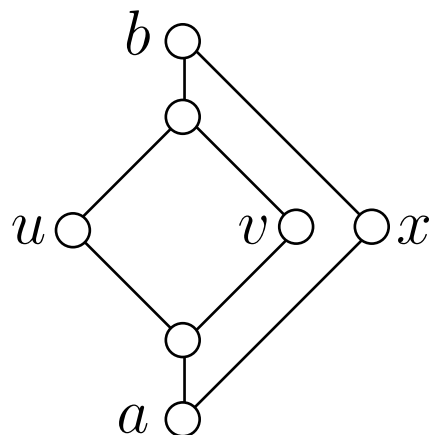
Making two elements perspective.

We are given elements $a \leq u, v \leq b$ in P such that

$$f_{\Theta_P}(a, u) = f_{\Theta_P}(a, v); \quad f_{\Theta_P}(u, b) = f_{\Theta_P}(v, b).$$

We put $Q = P \cup \{x\}$, where x freely adjoined such that

$$x \vee u = x \vee v = b; \quad x \wedge u = x \wedge v = a.$$



In particular, this forces the relations

$$\Theta_Q(a, u) = \Theta_Q(a, v); \quad \Theta_Q(u, b) = \Theta_Q(v, b).$$

Although $e_{P,Q}$ is no longer one-to-one, one can still find g , provided S is injective.

Identifying two congruences.

We are given elements $u, v \in [a, b]$ in P such that

$$f \Theta_P(a, u) = f \Theta_P(a, v).$$

By using E. T. Schmidt's $M_3[D]$ construction, we find an extension Q with elements u_i, v_i such that $u = u_0 \oplus u_1, v = v_0 \oplus v_1$ (within $[a, b]$), and

$$f \Theta_P(a, u_i) = f \Theta_P(a, v_i); \quad f \Theta_P(u_i, b) = f \Theta_P(v_i, b),$$

for all $i < 2$. Then apply the step above to u_0, v_0 , then to u_1, v_1 . Get an extension Q where

$$\Theta_Q(a, u) = \Theta_Q(a, v),$$

“free enough” for f to be factored through $\text{Con}_c Q$. (This also needs S to be injective.)

Adding one element in the image. We are given $a \in S \setminus \{0\}$. Pick $o \in P$, put $Q = P \cup \{x\}$, with x freely adjoined such that $o < x$. Extend f so that $g(\Theta_Q(o, x)) = a$.

This way, the new element $a \in S$ is forced into the image of f .

Injectivity (wrt. cofinal embeddings) of S is equivalent to

- (i) S is distributive;
- (ii) every bounded subset of S has a supremum;
- (iii) $a \vee \bigwedge X = \bigwedge(a \vee X)$, for $a \in S$ and $X \subseteq S$ nonempty and downward directed.

If (i)–(iii) above are satisfied, we say that S is *conditionally co-Brouwerian*, abbreviated CCB, and then we can iterate all the previously described steps transfinitely many times.

Theorem 26 (F. Wehrung, 2002) *For every CCB semilattice S and every partial lattice P , every $\langle \vee, 0 \rangle$ -homomorphism $f: \text{Con}_c P \rightarrow S$ can be lifted. Furthermore, S has 2-CLP.*

By using 2-ladders, we obtain the following.

Corollary 27 *Every direct limit of at most \aleph_1 CCB semilattices is isomorphic to $\text{Con}_c L$, for some relatively complemented lattice L with zero.*

And 1-CLP?

There are countable counterexamples where $f: \text{Con}_c P \rightarrow S$ cannot be lifted, for example for $S = \mathbb{Q}^+$. However, for S a lattice, something special happens.

The proof of Theorem 26 outlined above constructs a stepwise enlargement of the partial lattice P we are starting from. If we start with a lattice P , the next step (e.g., $Q = P \cup \{x\}$) is, in general, a partial lattice.

However, the partial lattices P and the maps $f: \text{Con}_c P \rightarrow S$ obtained in the process are quite special. View f as a S^d -valued structure on P , via

$$\|x \leq y\| = f\Theta^+(x, y).$$

Then, for example, $\|x \leq y\| \wedge \|y \leq z\| \leq \|x \leq z\|$.

We say that f is *balanced*, if $\|\theta\| = 1$, whenever θ is a sentence “expressing that joins and meets of ideals of P need only finitely many steps to be computed”.

In the inductive construction above, all the steps are balanced. The successive extensions of f can be calculated by explicit formulas, involving Boolean values.

We obtain the following.

Theorem 28 (F. Wehrung, 2003) *Every distributive lattice with zero has 1-CLP. In fact, it has “2-CLP for the bottom a finite lattice”.*

As a consequence of the methods, we obtain the following.

Corollary 29 *Every lattice L such that $\text{Con}_c L$ is a lattice admits a relatively complemented, congruence-preserving extension.*

This was first proved for $\text{Con}_c L$ finite by G. Grätzer and E. T. Schmidt in 1999.

The converse of Theorem 28 also holds (J. Tůma and F. Wehrung, 2002).

Theorem 30 *Every distributive $\langle \vee, 0 \rangle$ -semilattice with 1-CLP is a lattice.*

Hence, a distributive $\langle \vee, 0 \rangle$ -semilattice satisfies 1-CLP iff S is a lattice.

The smallest known counterexample of non-liftable map $f: \text{Con}_c K \rightarrow S$ satisfies $|S| = \aleph_0$ and $|K| = \aleph_1$.

Problem 3 *Let K be a countable lattice, let S be a countable $\langle \vee, 0 \rangle$ -semilattice. Can every $\langle \vee, 0 \rangle$ -homomorphism $f: \text{Con}_c K \rightarrow S$ be lifted?*

Given:

$$\text{Con}_c K \xrightarrow{f} S$$

Wanted:

$$\begin{array}{c} L \\ \uparrow f \\ K \end{array}$$

$$\begin{array}{ccc} & \text{Con}_c L & \\ & \uparrow & \searrow \varepsilon \cong \\ \text{Con}_c f & & \\ & \text{Con}_c K & \xrightarrow{f} S \end{array}$$

About representing semilattices (not maps), this yields the following modest improvement of Schmidt's representability result of distributive lattices with zero.

Corollary 31 *Every direct limit $S = \varinjlim_{n < \omega} D_n$, where the D_n -s are distributive lattices and the transition maps are $\langle \vee, 0 \rangle$ -homomorphisms, is isomorphic to $\text{Con}_c L$, for some relatively complemented lattice L with zero.*

In particular, every countable directed union $S = \bigcup_{n < \omega} D_n$, the D_n -s distributive lattices, is isomorphic to $\text{Con}_c L$, for some relatively complemented lattice L with zero.

Problem 4 *How about directed unions of the form $\bigcup_{\alpha < \omega_1} D_\alpha$, where the D_α -s are distributive 0-lattices?*

All known "uniform refinement properties" are not enough to solve this.

All this leads us to the following disappointing observation.

The URP barrier.

Proposition 32 *All existing representability results yield lattices L for which $\text{Con}_c L$ satisfies WURP.*

For example, no “representability theorem” is sufficient to yield the already represented $\text{Con}_c F_L(\omega_2)$!

This suggests the idea to start with lattices with complicated congruence lattices—for example, free lattices, then try to enlarge them to obtain tailor-made congruence lattices.

The key gadgets for such constructions are ideal/filter gluing plus the following.

The $M_3\langle L \rangle$ construction (G. Grätzer and F. Wehrung, 1999).

For a lattice L , let $M_3\langle L \rangle$ consist of all elements of L^3 of the form

$$\langle y \wedge z, x \wedge z, x \wedge y \rangle, \text{ for } x, y, z \in L.$$

It is a closure system in L^3 , thus a lattice.

Theorem 33 *The lattice $M_3\langle L \rangle$ is a congruence-preserving extension of L , via the map $x \mapsto \langle x, x, x \rangle$.*

Using this, G. Grätzer and E. T. Schmidt proved in 2001 the following.

Theorem 34 *Every lattice admits a regular congruence-preserving extension.*

A lattice L is *regular*, if any two congruences of L that share a congruence class are equal. In particular, every compact congruence of L is principal. Therefore,

If CLP can be solved positively, then it can be solved positively with lattices in which every compact congruence is principal.

Other uses of gluing and the $M_3\langle L \rangle$ construction yield the following.

The Strong Independence Theorem for congruence lattices and automorphism groups of lattices (G. Grätzer and F. Wehrung, 2000). *For every nontrivial lattice L_C and every lattice L_A , there exists a lattice L , which is a congruence-preserving extension of L_C and an automorphism-preserving extension of L_A .*

The construction uses a generalization of the $M_3\langle L \rangle$ construction, called *box product* of lattices. Furthermore, it may yield

$$|L| > \max\{|L_A|, |L_C|, \aleph_0\}.$$

But here, the congruence lattice does not move
...

Second magic wand Theorem (G. Grätzer, M. Greenberg, and E. T. Schmidt, 2002) *Let K be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of K , and let φ be a homomorphism of $[a, b]$ onto $[c, d]$. There exists a bounded convex extension L of K such that a congruence θ of K has an extension to L iff $x \equiv y \pmod{\theta}$ implies that $\varphi(x) \equiv \varphi(y) \pmod{\theta}$, for $a \leq x \leq y \leq b$, in which case, θ has a unique extension to L .*

So, here, $\text{Con } L$ is isomorphic to $\text{Con}^\varphi K$, the lattice of all φ -preserving congruences of K . In particular, $\text{Con}_c L$ is the image of $\text{Con}_c K$ under a *distributive homomorphism* (i.e., a homomorphism whose kernel is the union of kernels of algebraic closure operators).

In particular, if $\text{Con}_c K$ has WURP, then so does $\text{Con}_c L$.

Thus the following is of importance.

Problem 5 *Let K be a lattice, let S be a distributive $\langle \vee, 0 \rangle$ -semilattice. Can every surjective, distributive homomorphism $f: \text{Con}_c K \twoheadrightarrow S$ be lifted?*

A precursor to this problem is Schmidt's Lemma (1968), that states that any distributive image of a generalized Boolean lattice is representable.

Can CLP be solved with a magical formula?

P. Pudlák found such a magical formula to represent distributive *lattices* with zero. For semilattices, a natural question is to ask whether there exists a functor Ψ from distributive $\langle \vee, 0 \rangle$ -semilattices to lattices (“ Con_C^{-1} ”) such that $\text{Con}_C \circ \Psi$ is naturally equivalent to the identity.

Proposition 35 *There is no such functor.*

(see J. Tůma and F. Wehrung, 2001).

This is easy to prove. If there would be such a functor Ψ , try to lift the map $e_\alpha: \mathbf{2}^\alpha \rightarrow \mathbf{2}$, $x \mapsto 1$ if $x \neq 0$, 0 otherwise (α an ordinal). It separates 0, thus $\Psi(e_\alpha): \Psi(\mathbf{2}^\alpha) \rightarrow \Psi(\mathbf{2})$ is a lattice embedding. But $\Psi(\mathbf{2}^\alpha)$ is arbitrarily large.

However, e_α is not a semilattice embedding ...

And for semilattice embeddings?

Nobody knows (whether there is a “ Con_c^{-1} ” functor, on distributive $\langle \vee, 0 \rangle$ -semilattices and their embeddings)!!!

Even worse, the counterexamples not representable with permutable congruences are distributive semilattices of a very particular sort.

Definition 36 A $\langle \vee, 0 \rangle$ -semilattice S is ultraboolean, if $S = \varinjlim_{i \in I} S_i$, where all the S_i -s are finite Boolean and all transition maps are $\langle \vee, 0 \rangle$ -embeddings.

Equivalently, every finite subset of S is contained in some finite Boolean $\langle \vee, 0 \rangle$ -subsemilattice of S .

Every ultraboolean semilattice is distributive. However, $\mathbf{3}$ is not ultraboolean.

Problem 6 Is there any “ Con_c^{-1} ” functor on the category of ultraboolean semilattices?

In order to reduce the problem from arbitrary distributive semilattices to ultraboolean ones, we may ask the following.

Problem 7 *Is every distributive $\langle \vee, 0 \rangle$ -semilattice the image of an ultraboolean semilattice under some distributive homomorphism?*

Problem 7 has a positive answer in the *countable* case, by using a closure operator. Also for semilattices of size \aleph_1 we get a positive answer, because of A. P. Huhn's results.

Problem 8 *Is every distributive $\langle \vee, 0 \rangle$ -semilattice a retract of some ultraboolean semilattice? Can this be made functorial?*