TO THE MEMORY OF GEORGE KAC

# IDEAS THAT WILL OUTLAST US

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Translated into English by Nataliya Markova

## PREFACE

The period I knew Georgiy Isaakovitch Kac, a remarkable mathematician and remarkable person, lasted 1968 to 1978. These were the last years of his life. To our great sorrow, he suddenly died of a heart attack on May 20, 1978, in the prime of his talent and vitality. Certainly, his family and friends knew his personality better than I did. For example, B.I. Khatset wrote that his modesty and generosity had earned him a mock nickname of Pierre Bezukhov (one of the principal characters of Leo Tolstoy's *War and Peace*), and as you will see below, he was like this in mathematics, too. But probably only V.G. Paljutkin, who, like me, had worked under G.K.'s guidance, knows his ways in creative work better than myself. Besides, I had an opportunity to observe vigorous development of his ideas after his death, having spent considerable time at research centers where the follow-up work took place. This is why you will find here not just memories of events and sensations of those days, but also reflections on his mathematical ideas, their genesis, their evolution and impact on other researchers. Of course, this is not a scientific paper and is neither rigorous nor exhaustive. Nevertheless, it will seem more understandable to those who are to a certain extent familiar with algebra and analysis. Things presented here are the exposition of my personal views. I am writing mainly about events that I had experienced myself. The number of works that had drawn on Kac's ideas and results obtained by him, included to hundreds of titles even back in 1992 (see book [22]), but these notes refer only to the publications that in my opinion are most helpful to those interested in Kac's mathematical legacy. I am mostly describing my personal views and experiences.

This text is an English translation of the original text for students published in Russian in the summer of 2004 on the occasion of George Kac's 80th anniversary ("In the World of Mathematics", Kiev University). The bibliography keeps the initial Cyrillic alphabetical order, but references are given to the English translations of the papers. The only exception is [14]; this important paper has never been translated into English.

I am grateful to Igor L. Markov for many helpful remarks concerning the translation of this text into English.

## KAC ALGEBRAS

In the Fall of 1968 I started to attend George Kac's seminar on operator rings at the Institute of Mathematics of the Ukrainian Academy of Sciences. I still have memories, though sketchy, of his lectures on works by Glimm, Dixmier and Douady, and Fell. G.K. was a remarkable speaker; his manner of setting out the material was clear and rigorous, and even more importantly, lively and accessible. Nothing I had heard before could match it; and long after his death only a handful of speakers left a comparable impression.

Up to the summer of 1969 our relationships were no more than polite greetings when we met. We came to closer know each other only after my graduation from the university, when I entered the Kiev Aviation Engineering Military College, where Kac served as professor of mathematics. Despite graduating with honors, it was impossible for me to formally enroll as a Ph.D. student in those years, at the height of anti-Semitism. It was for the same reason that G.K. could not work either at the University, or within the system of the Academy of Sciences. But Yu.M. Berezanski had helped him to set up his seminar at the Institute of Mathematics. In spite of these difficulties, over the following few years, I was fortunate to pursue an "informal" Ph.D. program under the supervision of G.K., that not has only guided my scientific career, but also determined my further life path. I cannot fail to mention that in the same period M.L. Gorbachuk was advising me in an equally informal Ph.D. program, that ended in a doctoral dissertation in 1975. I feel a deep gratitude to him for the teachings and also for having the will to give his backing to a Jew, which was no small thing in the USSR those days.

For a start G.K. assigned me to recite to him the book "Lie algebras and Lie groups" by J.-P. Serre, and then offered the choice of two research topics. The first one was to further develop his work with F.A. Berezin [3], then in print, now considered as one of the pioneering works of a trend that later evolved into a full-fledged branch of mathematics, jokingly referred to as supermathematics. It is dedicated to the study of mathematical structures (superalgebras, Lie superalgebras, supermanifolds, etc.) graded by some group, in the simplest case by the group Z/2Z. The arrival of these structures was motivated by quantum mechanics, where two kinds of particles, fermions and bosons, are governed by totally different statistical laws. Both Kac and Berezin had a strong theoretical physical background, so it was for good reason that they were pioneers in this new area. G.K. got his candidate of science degree (an equivalent of Ph.D.) in 1950 under the supervision of N.N. Bogolyubov with the thesis on the correlation theory of electron gas.

Nevertheless I missed my chance to become a “supermathematician” and chose another topic, the so-called *ring groups*, introduced by G.K. around 1960. Let me remind of the circumstances that gave rise to this theory.

Let G be a commutative locally compact group and let $\underline{G}$ be the group of its unitary continuous characters, which is also commutative and locally compact and is called a dual group of G. Then it turns out that the group dual to $\underline{G}$ is isomorphic to G, which is the duality principle of L.S. Pontryagin. However, this beautiful theory breaks down if the group G becomes non-commutative, even when it is finite, since the characters of such a group are too few and do not contain all the information about the group. A natural way to save the duality theory would be to replace the group’s characters with its irreducible unitary representations (recall that irreducible representations of commutative groups are just characters). Indeed, T. Tannaka showed in 1938 that it was possible to restore a compact group up to isomorphism from a full collection of its representations, while in 1949 M.G. Krein gave a full axiomatic description of such dual object for a compact group. Nevertheless, the mathematical structure of this dual object (a block-algebra) was different from that of the group, thus breaking the symmetry of the duality. Later on, such non-symmetric duality theory was developed by W.F. Stinespring (1959) for unimodular groups and by N. Tatsuuma (1965-66) for any locally compact groups.

The Editorial Board of the “Matematika” collection of translations asked G.K. to translate into Russian the above-mentioned paper by Stinespring. During his work on the translation he came up with a brilliant idea to construct a new category, whose objects he called ring groups, which would contain both groups and their duals with a symmetric duality principle acting within. The latter required that the mathematical structure of the original object and that of its dual be the same, as in the case of Potryagin's duality. More precisely, G.K. proposed the following:

Let **A** be a commutative algebra of functions on a unimodular group **G**, that are measurable and essentially-bounded with respect to the invariant measure on **G**. The group operation of multiplication allows to build a homomorphism **Γ** of algebras, that sends each function **f(x)** from **A** to a function of two variables **f(xy)**, i.e., to an element of a tensor product of two copies of **A**; **Γ** is called *comultiplication*. The operation of inversion in the group **G** can be "encoded" in the form of an anti-isomorphism **S** that maps **A** onto itself (referred to as *antipode*), and the invariant group measure can be "encoded" in the form of a positive linear functional **m** on **A** (i.e., the integral over this measure; this functional, which is also called *invariant measure*, can take infinite values on certain functions). Thus, the entire information about the group **G** may be expressed in terms of the collection (**A,Γ,S,m**), where **A** is an algebra (more exactly, a von Neumann algebra) and **Γ,S,m** are maps that satisfy certain conditions. Such a collection, where algebra **A** is not necessarily commutative, is said to be a ring group, while algebras of functions on groups exactly correspond to ring groups with commutative algebras. In purely algebraic sense, ring groups proved to be nothing other than *Hopf algebras*, that arose earlier in topology. G.K. was not aware of the existence of Hopf algebras and reinvented them when he introduced ring groups.

The dual object of an ordinary group can also be described as a ring group with cocommutative comultiplication (i.e., one that is stable with respect to the permutation of factors in a tensor product). In this case, the algebra **A** is generated by shift operators **L(g)** on the group (where **g** is an element of the group **G**), or equivalently by convolution operators **L(F)**, where **F** is a continuous integrable function on the group. Comultiplication sends **L(g)** onto the tensor product of two copies of **L(g)**, the antipode **S** sends **L(g)** onto the adjoint operator, the value of the functional **m** on the operator **L(F)** is the evaluation of the function **F** at the unit of **G**. Lastly, G.K. proposed a construction allowing to build, out of a given ring group (which is not necessarily commutative or cocommutative), its dual. Applying this construction twice, we get an object which is isomorphic to the original one, as in the case of Pontryagin duality. These results were first announced in Soviet Math. Dokl. in 1961 and then published in the comprehensive paper [11]. That work relied on the techniques from I. Segal's theory of *traces* on von Neumann algebras (a trace is a positive linear central functional, i.e., such that **m(ab)=m(ba)**; G.K. had also translated Segal's paper on traces into Russian in the "Matematika" collection). In terms of groups, it represented the construction of a symmetric duality theory for arbitrary unimodular groups. An important open problem that had already been formulated in G.K.'s habilitation dissertation (Moscow State University, 1963), was to generalize this theory so that it would cover all locally compact groups. That was the very problem suggested to me as a research topic in the early 1970.

Fairly quickly I realized what exactly hindered that generalization. In the theory of ring groups, the functional **m** had been a trace, so to generalize the theory one had to learn how to handle non necessarily central positive functionals on operator algebras that can take infinite values (called *weights*). Besides, in his theory, G.K. made systematic use of close ties between traces and so-called *Hilbert algebras*, so that one had, in parallel with generalizing the theory of traces, to develop an appropriate generalization of the theory of Hilbert algebras. The idea appealed to G.K., and we started to actively work on that. However, papers [16] and [20], where the above problems had been solved, appeared shortly, one after another; we had no difficulty to understand them since we had already went halfway.

Now the coveted target came within reach, but we had to hurry, since we were not alone in our pursuit. First, by that time M. Takesaki had already written a paper on generalization of ring groups; in addition he had mastered all the necessary techniques. It is still a mystery to me why he had not come first in that race, being a leading mathematician with a number of brilliant results to his credit. G.K. was also convinced that J. Dixmier saw the same objective. I remember him saying "We must hurry, Leonid, since I'm sure that Dixmier has put someone to tackle this problem." As it turned out later, he was absolutely right. But even in that stressful situation his integrity did not fail him: since he considered that it was me who had suggested basic ideas for a possible solution, he decided to allow me an opportunity to complete the solution by myself, thus becoming the sole author, while it was he who had formulated the problem and made strong efforts to adequately prepare me for solving it. Besides, we used to talk about related subjects, since he liked me to come to the lecture room by the end of his undergraduate lectures and to walk with him on foot from the Uritski Square, where the Military College was situated, past the railway station to his home on the Bolshaya Podvalnaya Street (the streets' names are of the period). At times these discussions continued in his apartment. We also had a lot of talks in lecture rooms, at a desk or blackboard. Later G.K.'s seminar on operator rings moved to the House of the Promotion of Science and Engineering, and after G.K.'s death Yu.L. Daletski took over the seminar.

Working on my own was rather risky, because I had not fully mastered the necessary techniques yet, and to do that I needed time. To make the matters worse, I was deeply upset by the death of my father in August 1971. Finally, seeing that I was not making much progress, G.K. understood that we could well loose the race, and took over. He quickly got through a couple of places that had baffled me, and with his support I started making much more progress. In a concerted effort, we had fairly rapidly completed a draft. Even now G.K. remained true to himself; he continued to view me as the originator of the basic ideas of the solution and suggested that I first publish part of the solution on my own, and only after that we publish the entire solution, which we did --- the note [6] was submitted earlier than both papers [7].

Then we were in for an ordeal. There appeared a long paper [21] by Takesaki, in which he took one more step towards the generalization of ring groups. The paper was unavailable in Kiev, while its review in "Реферативный Журнал" (the Soviet analogue of *Mathematical Reviews*) reported the construction of a complete duality theory which generalized the theory of G.I. Kac's ring groups and covered arbitrary locally compact groups. Given the author's reputation and the title of the paper, there was little doubt that we had lost. G.K., upset, dropped all his work and rushed to Moscow to read it for himself (prior to that, he made a point of visiting Moscow several times a year in order to keep track of publications that were unavailable in Kiev and to socialize with colleagues, such as M.A. Naimark, F.A. Berezin, A.A. Kirillov, A.I. Shtern and others). I felt desperate. And then G.K. came back with a photocopy of the Takesaki's paper, and it became clear that the reviewer was mistaken and the paper's results had not achieved our (and not only our) goal.

We completed and published our two papers. But virtually at the same time, out were the papers by M. Enock and J.-M. Schwartz who worked under Dixmier's supervision, containing equivalent results, although using a somewhat different technique. As later Michel Enock told me, Dixmier had also urged them to hurry, explaining that besides Takesaki, there had to be someone in Kiev, working on the same problem with Kac. Those days, as later, the French colleagues used to immediately send us their preprints and newly published papers, while we could not respond to them at all, since we worked at a Military College and, bound by secrecy regulations, were not allowed to communicate with foreigners. For example, in 1995 we were invited to participate in a conference in Marseille, that was dedicated to the subject of our studies. From the outset, G.K. said that we could not go, but I rashly showed the invitation at the so-called First Department of the College. I was lucky to get away with it; the KGB men explained that if I wanted to continue my work at the College, they would act as though I had not shown them the invitation.

In view of the obviously fundamental role of G.K. in the discovery of the new mathematical objects that we had introduced, the suggestion of Enock and Schwartz to name them *Kac algebras* was highly appropriate (the reader is probably aware of the existence of Kac-Moody algebras, named after Victor Kac and having absolutely different nature). The first work referring to that term appeared in 1974; I learned about it from G.K. himself. His eyes were very expressive, and as he was talking about that paper, one could see how much he enjoyed the news. Of course, being a man of great modesty, he never actually pronounced the term Kac algebra. Today the entire book [22] is dedicated to the theory of Kac algebras.

Of course, Kac algebras have been invented with a view to apply them to the solution of various problems, and not just for the sake of beauty. G.K. used to say that ring groups had to be regarded in the same way as the ordinary groups that they generalize, while their application areas might be wider than those of ordinary groups, and the results might be more complete, which later proved to be the case. But to bring ring groups into effective use one had first to gain an understanding of their structure, examples and properties. After all, they were not groups, but rather a far-reaching generalization. Back in the early sixties G.K. had distinguished and started to explore special classes of ring groups, namely *compact, discrete, finite ring groups* and their representations. Subsequently he carried on that work in collaboration with V.G. Palyutkin.

G.K.'s habilitation dissertation contained a long list of problems to be solved; some of them have been solved since (one of them was examined above), others have remained unsolved to this day. One of these open problems is based on the fact that the existence of an invariant measure is explicitly contained in the definition of Kac algebra and not derived from other axioms, as in the case of ordinary groups (there is no difficulty in proving the uniqueness of this measure). These axioms are justified by Weil's theorem which says that the existence of an invariant measure on a group entails the existence of a locally compact topology on it. Combined with the Haar theorem (i.e., the inverse statement), it means that the existence of an invariant measure and the existence of a locally compact topology are equivalent. However, it would seem more natural to define Kac algebras only in algebraic and topological terms and *to prove* the existence of an invariant measure, thereby generalizing the Haar theorem for ordinary groups. For the foregoing special classes of Kac algebras this difficulty was overcome in [15], [19].

In particular, finite Kac algebras [15] are finite-dimensional semisimple Hopf *-algebras over the field of complex numbers. Instead of an invariant measure the axiomatics contains *a counit*, i.e., the character that is in a special way connected with comultiplication and antipode and is the analogue of the unit in an ordinary group. The existence and uniqueness of an invariant measure represent a theorem. Morphisms and subsequently, subobjects, factor-objects, etc. are defined in a natural way. The work [15] is written very clearly. I remember it to have deeply impressed me, and even now I suggest it to young mathematicians as their first reading on the topic. This kind of experience proved especially successful with Dmitri Nikshych in 1995 and 1996. The paper stirred his interest, and he has worked on interesting generalizations of Kac algebras since then. The axiomatics of the compact Kac algebras is presented in [19] in the same vein; this paper contains the theorem of existence of an invariant measure.

In his habilitation thesis and paper [13], G.K. has shown that a number of classical results on finite groups can be extended to finite Kac algebras. In particular, he has obtained the analogue of the Lagrange theorem stating that the order of a subgroup divides the order of the group. Later on, a stronger statement was established by V.D. Nichols and M.B. Zoeller for any finite-dimensional Hopf algebra. As for finite groups, the only Kac algebra of a simple dimension **p** is the cyclic group with **p** elements. This result has been more than once generalized in recent works by various authors. G.K. has also proved that for any irreducible representation of a finite Kac algebra there exists a basis, in which the matrix elements of this representation are algebraic integers.

As already noted, commutative Kac algebras correspond to ordinary groups, while cocommutative Kac algebras correspond to objects dual to ordinary groups. Of special interest are examples of Kac algebras that do not belong to these two classes, they are called *nontrivial.* First such examples were built by G.I. Kac and V.G. Paljutkin, see [12], [14], [15]. Below I will try to explain the motivations behind them; so far I will only point out that, as noted by V.G. Drinfeld in his fundamental work [10], these examples turned out to be historically the first examples of the so called *quantum groups* which have numerous important applications.

Much as ordinary groups are of interest first and foremost as set transformation groups, Kac algebras can also act, but on algebras instead of sets. G.K. repeatedly voiced an opinion, which is now generally accepted, that algebras are noncommutative counterparts of sets, in the same way as ring groups are noncommutative counterparts of groups. He suggested a definition of *a ring group action* on an algebra and a construction of the cross-product of these objects. There was a series of works by Enock and Schwartz dedicated to that type of constructions and related results.

Now let us get back to Kiev in mid-seventies. When discussing the choice of new problems, G.K. strongly advocated an in-depth study of finite ring groups and their representations, using known results about finite groups as a base, in the vein of his works. Another option was the search for new nontrivial examples of Kac algebras, but that turned out to be an uphill task, and only more than 20 years later, in an absolutely new setting, both in practical and mathematical sense, I learned how to construct them on a systematic basis. But in those days analytical aspects of the theory seemed closer to me than algebraic ones, though the possibilities in that direction looked limited.

Gradually, a decision formed in my mind to take up such generalization of Kac algebras that would cover so called generalized shift operators, or hypergroups. Earlier generalized shift operators had been intensively handled by such leading Soviet mathematicians as Yu.M. Berezanski, S.G. Krein and B.M. Levitan. This line of research offered challenging analytical problems and one could hope for useful applications and concrete examples. I was disappointed with G.K.'s lukewarm reaction, despite the fact that such problems were mentioned in his habilitation dissertation and in his work [11]. He said that it had appealed to him before, but later he realized that in algebraic terms such theory would be somewhat deficient. Indeed, in the Kac algebra theory comultiplication **Γ** sends a product of elements of algebra **A** again into a product, but with a far weaker positivity condition placed on it, certain essential properties of Kac algebras do not hold in this more general setting. Still, the fact that due to weaker constraints the theory became far richer in applications, seemed of importance to me. I was fascinated by this idea, which later became the subject of my habilitation dissertation defended in 1992. In the eighties and nineties, encouraged by Yu.M. Berezanski, I wrote a series of papers, some of them in collaboration with A.A. Kalyuzhnyi, Yu.A. Chapovski and G.B. Podkolzin. But in the mid-seventies, to my great sorrow, our active collaboration with G.K. was practically over. Still we continued to see each other and discuss mathematical and non-mathematical events. Since then, I had always kept track of the Kac algebra theory, that was my first love in Mathematics, but it was not until mid-nineties that I genuinely returned to it.

## QUANTUM GROUPS

Let us now turn to the events associated with the evolution of Georgiy Isaakovitch's ideas after his death. From the experience of working with Kac algebras one could see that while having solved the problems that had brought about their creation, their range of application was not wide enough, which in fact was the reason for me to take up a generalization. But the genuine breakthrough in the understanding of the direction to take to advance this range of ideas, occurred in the mid-eighties, when V.G. Drinfeld along with some other mathematicians discovered the world of quantum groups [10]. In purely algebraic terms, the only difference between a quantum group and a Kac algebra lies in the fact that squared antipode is not necessarily an identity --- a seemingly small matter, but the one that made a fundamental difference. Not to mention the fact that the algebra in question is not necessarily semi-simple, and the ground field might not be a field of complex numbers. Many important specific examples and applications appeared, in particular in theoretical physics and topology. As a matter of fact, it was the in-depth analysis of mathematical models of quantum scattering theory by researchers from L.D. Faddeev's group in Sankt-Petersburg that resulted in the invention of quantum groups. As Alain Connes put it in the preface to the book [22], Kac algebras have proved "not sufficiently unimodular" to cover new applications and therefore the necessity arose for quantum groups. Note that Kac algebras are in their own right "non-unimodular" generalization of ring groups!

In addition to a host of purely algebraic works (in which, for example, deformations of virtually all classical Lie groups have been built) a number of quantum groups emerged in which topology also played an important role. In particular, S.L. Woronowicz built in [9] the theory of compact quantum groups that started from the theorem of existence of invariant measure and then carried on a systematic study of irreducible representations and their matrix elements. As you may remember, in the theory of compact Kac algebras such Haar theorem had been proven by V.G. Paljutkin much earlier. Woronowicz also built a number of concrete examples of quantum groups --- both "algebras of continuous functions" and their dual objects. In so doing, he had to overcome numerous functional analytical difficulties, inherent in each specific case. This work highlighted the challenges in extending the Kac algebra theory to capture all interesting examples while keeping the most essential features, such as the beauty and symmetry of the previous theory.

An important step to the construction of the new theory had been made by S. Baaj and G. Skandalis [1]. It was the already cited article by Stinespring that had highlighted the essential role played in the duality theory of unimodular groups by the unitary **W** that sends a function of two variables **f**(**x**,**y**) into **f**(**x**,**xy**). G.K. had built an analogue of this operator for arbitrary ring group and shown that its fundamental property is the so-called *pentagonal relation* $\mathbf{W}_{23}\,\mathbf{W}_{13}\,\mathbf{W}_{12} = \mathbf{W}_{12}\,\mathbf{W}_{23}$ , where the indices show which two of the three components of the element belonging to the tensor cube of algebra A are distinct from 1. He was the first to have written out this important relation and to become aware of its key role in duality theory. Later on, G.K. and myself, as well as Enock and Scwhartz, made full use of this observation in the process of construction of non-unimodular Kac algebras. But Baaj and Skandalis took it even further, pointing out that a pentagonal-relation-satisfying unitary operator in itself, along with its certain regularity conditions, allows to build two operator algebras in duality, each of them carrying a more general structure than that of a Kac algebra. They named this object a multiplicative unitary.

After the end of the Soviet era our life took a new turn. A friend of mine, Dmitri Gurevich, was able to visit France; he passed my regards and several publications to Michel Enock who, apparently astonished by the fact that I really exist, in return sent me several kilograms of his works for all the years past. He could not know that I had gotten everything he had sent me since not a single response came back from behind the Iron Curtain (the envelopes often had traces of being open and re-sealed, certainly by the KGB, but who would be concerned with mathematical articles?). We started to correspond, and soon Michel Enock came to Kiev. It was then, in the spring of 1991, that he showed me the preprint of the work [1]. One of sentimental events of his stay was a visit to G.K.'s tomb at the Gostomelskoye cemetery.

In the spring of 1992 it was my turn to come to Paris on Michel's invitation. Brimming with impressions as I was, one episode etched vividly in my memory: Adrian Ocneanu said at the beginning of his talk that he was sincerely happy to greet in person a representative of George Kac's school. He did not know that G.K. could never have formal graduate students, and finally, to my knowledge, all his school consisted of two and a half students, V.G. Paljutkin, myself and V. Zhuk (the latter had worked with G.K. in the mid-seventies, but his research was not fruitful enough). Paljutkin and Zhuk were formal graduate students of Yu.M. Berezanski. In fact, receiving tokens of respect in Paris, I was well aware of the fact that they concerned rather G.K. than me. How unfair that he himself had not enjoyed even a small part of the recognition he had so much deserved! Besides the foregoing difficulties of his life, I would like to point out that his scientific results, being far ahead of their time, had not been properly appreciated by some leading mathematicians. I can cite as examples L.S. Pontryagin, whose duality principle had been so brilliantly extended by G.K., and I.M. Gel'fand, who had been rather critical of G.K.'s works. In the summer of 1983, Gel'fand chided me, trying to persuade me to take up a different subject, "Do not hide your light under a bushel as your teacher Kac did!" All that just a short time before the discovery of quantum groups, for which V.G. Drinfeld received the Fields Medal (the analogue of the Nobel Prize for mathematicians) and which had the Kac algebra theory as their direct predecessor, see [10]!

Gradually the idea arose to get back to constructing non-trivial examples of Kac algebras and quantum groups. Although Woronowicz and others had already built plenty of concrete examples of quantum groups, each time it was in a sense custom work that involved overcoming major functional analytic difficulties. In my opinion, it was desirable to have a construction which would allow to get many different examples by a unified technique. For example, Drinfeld proposed a purely algebraic way to change comultiplication **Γ** and antipode **S** without changing the algebra **A** of a given quantum group in order to get a new quantum group. Applying this construction, which is called *twisting*, to the object that is dual to an ordinary group one could hope to come up with interesting examples of quantum groups. I started to figure out the way to develop analytical aspects of twisting.

Enock came back to Kiev in May 1994 for G.K.'s 70$^{th}$ anniversary. With the backing of M.L. Gorbachuk, the then president of the Kiev Mathematical Society, both of us gave presentations at the Society meeting about different facets of G.K.'s activities. Among other topics of discussion, there was the problem of examples of Kac algebras. By the autumn, already familiar with the results obtained by M. Rieffel and M. Landstad, I was figuring out how to adjust Drinfeld's twisting to our purposes and in the spring of 1995 in Paris, Enock and I finished the paper [23]. Later I extended and reinforced its results in [8]. The finite-dimensional aspect of this construction was the subject of intensive discussions in Kiev with D. Nikshych, who wrote an interesting paper on that occasion. These activities resulted, in addition to an abstract construction, in an entire series of new "quantizations" of the Heisenberg group, popular with physicists. All of them turned out to be more than just Kac algebras, they were unimodular ring groups in the sense of the very first definition by G.K. Besides, we got quantizations of classical series of finite groups --- symmetric, dihedral, quasiquaternionic groups and some others. Of special interest were Kac algebras obtained by Nikshych from alternated groups. The latter, as is well known, are simple groups (starting from number 5 in the series); the corresponding Kac algebras also proved to be simple in a certain natural sense, thereby giving a positive answer to the question of Victor Kac about the existence of such objects.

In the late nineties, a number of events occurred that brought progress on long-awaited generalization of the Kac algebra theory. Baaj and Skandalis, on the one hand, and Woronowicz, on the other, refined the understanding of regularity conditions to be placed on a multiplicative unitary so that it generate a pair of quantum groups possessing reasonable properties. A version of the theory of such quantum groups was announced by T. Masuda, I. Nakagami and S.L. Woronowicz. In the Belgian town of Leuven, A. Van Daele set up a small but active research group to discuss new examples of quantum groups and approaches to the construction of a general theory. In particular, for the class of quantum groups distinguished by Van Daele, axioms were formulated in a purely algebraic way and then all topological properties could be derived. It was rather convenient to study individual salient features of the future general theory on this "laboratory material". Besides, J. Kustermans explored in detail the properties of weights (i.e., linear positive functionals that can take infinite values) on C*-algebras. Weights had already been used in the early seventies in the construction of the non-unimodular Kac algebra theory, but now a more profound investigation was needed, since a stronger non-unimodularity was involved (I will give more details below).

Finally, a general theory of locally compact quantum groups was proposed in 1999 by Van Daele's students, J. Kustermans and S. Vaes, see [17]. It was as beautiful and symmetric as the Kac algebra theory was; without much exaggeration one can say that it had been modeled on the latter. More exactly, a locally compact quantum group in the von Neumann algebra version is the collection (**A**,**Γ**,**m**,**n**,), where algebra **A** and comultiplication **Γ** are the same as in the Kac algebra theory, and **m**, **n** are respectively left and right invariant weight on **A**. These axioms do not mention antipode but imply its existence and properties. And instead of one, two weights are present – here it is, the second non-unimodularity! It resembles an ordinary non-unimodular locally compact group that has two invariant measures, left and right. The structure of the theory resembles that of Kac algebras, especially when it comes to duality. Of course, technically it is far more complicated, in return it covers virtually all known examples of locally compact quantum groups.

Starting from September 1999, I spent a few months in Leuven, where I could familiarize myself with these things firsthand, thereupon I could see a possibility of bringing to life an idea originating in the magnificent *theory of extensions* of finite Kac algebras [12]. As indicated above, a commutative Kac algebra $\mathbf{K}_1$ is an algebra of functions on an ordinary group $\mathbf{G}_1$, and a cocommutative $\mathbf{K}_2$ is a dual object to the ordinary group $\mathbf{G}_2$. Given two such groups, and consequently, two corresponding Kac algebras, the question is whether it is possible to build a new Kac algebra **K** so that $\mathbf{K}_1$, **K** and $\mathbf{K}_2$ form an exact sequence. This means that $\mathbf{K}_1$ becomes a normal subalgebra of **K**, and $\mathbf{K}_2$ becomes the corresponding factor-algebra (here **K** is called an extension of $\mathbf{K}_1$ by means of $\mathbf{K}_2$). In [12] G.K. had given a comprehensive answer to this question in the case of finite groups: for the existence of such an extension, it is necessary and sufficient that groups $\mathbf{G}_1$ and $\mathbf{G}_2$ act on each other as on sets and that these actions be compatible in a special way. He had fully described the construction of all such extensions, which is nowadays called *bicrossed product*. It was exactly the idea underlying first non-trivial examples of Kac algebras in [12], [14], [15], even though the example in [14], where $\mathbf{G}_1$ and $\mathbf{G}_2$ were Lie groups, was not supported by the general extension theory.

Later on, this construction was rediscovered by various authors, among them M. Takeuchi and Sh. Majid. The latter addressed not only finite but topological groups as well, see book [18] and references therein. However, for arbitrary general locally compact groups, the mathematical nature of their extension might be rather complicated. For example, to be sure to obtain a Kac algebra as an extension, one had to impose quite strict constraints on $\mathbf{G}_1$ and $\mathbf{G}_2$, which is precisely what Majid had done. But now that we had a far wider category of locally compact quantum groups in hand, one might expect to make the most of the construction. I shared these ideas with Stefaan Vaes, and in a few months we had turned our plans into reality. The work [4] has fully lived to its promise.

As for concrete examples of Kac algebras and quantum groups, the paper [4] contained only a few of them, but our later work [5] featured a variety of low-dimensional Lie groups as $\mathbf{G}_1$ and $\mathbf{G}_2$ which led to many examples that we classified according to their properties. In particular, necessary and sufficient conditions were given, under which an extension was a Kac algebra. An example of a distinctly different quantum group with unexpected regularity properties, in which $\mathbf{G}_1$ and $\mathbf{G}_2$ are groups coming from number theory, was later built by Baaj, Skandalis and Vaes. They had also carried over to locally compact quantum groups another brilliant idea of G.K. from [12]. Namely, to describe non-equivalent extensions of quantum groups, G.K. had built a very interesting cohomology theory, that included the so-called Kac exact sequence (it must be clear to the reader that this terminology was introduced later, when the importance of this sequence in various problems became evident). Over recent years the Kac exact sequence has gained popularity among specialists in quantum groups and pure algebraists as well (A. Masuoka, P. Schauenburg, etc.)

I will conclude my notes with a story about Kac algebras "in action". As noted above, Kac algebras can act on non-commutative algebras similarly to how groups can act on sets. To this end, Ocneanu in the postface to the book [22] explained that Kac algebras must arise as non-commutative analogues of group symmetries in the theory of subfactors (the originator of this theory, V. Jones was awarded the Fields Medal). Indeed, in 1994 W. Szymanski and R. Longo independently of one another demonstrated that if

**N** is a so-called subfactor of depth 2 and of a finite index of factor **M**, while their relative commutant (i.e., the set of elements from **M** that commute with **N**), is trivial then there is necessarily a Kac algebra **K** acting on **M** such that **N** is a subalgebra of fixed points with respect to this action. An independent proof was given later by M.-C. David. The construction of the foregoing Kac algebra and its action was given explicitly. The converse is also true: given a Kac algebra and its action on a factor, one can build a corresponding subfactor with the properties listed above. This situation resembles the classical Galois theory, where **N** and **M** are fields, and **K** is a Galois group. One can show that this analogy is far-reaching.

This impressive result had triggered an avalanche of works. Among the most important was the work by M. Enock and R. Nest, who came up with a similar result for subfactors of infinite index. Naturally, in that case a Kac algebra had to be replaced by a locally compact quantum group. Another important generalization can be obtained if the triviality condition of the relative commutant is dropped. In that case a Kac algebra must be replaced by a so-called quantum groupoid, a mathematical structure invented by theoretical physicists. Its principal distinction from Kac algebra lies in the fact that comultiplication **Γ** is not necessarily a unital map; a quantum groupoid is a generalization of an ordinary groupoid in the same vein as a Kac algebra is a generalization of an ordinary group. There exist many works, including those by Nikshych and myself, that extend various results from the Kac algebra theory to quantum groupoids. The references and historical notes on quantum groups and quantum groupoids can be found in the editor's introduction to the book cited in [5].

## POSTFACE

These notes have been written in January to March 2004 in the French town of Caen, far away in space and time from a Kiev of the early seventies and my first steps in Mathematics under the supervision of Georgiy Isaakovich, so it is no surprise that my memories are tinged with nostalgia. But first and foremost I wanted to show the powerful influence of his works, not numerous as they were, on the progress of a wide area of mathematics. It has been more than 26 years since he was gone, but in scores of works published in 2003 and 2004, you will effortlessly discover clear evidence of their origin from his ideas. Indeed, as Pushkin has put it, "I have built a monument not wrought by hands…"

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