A Mordell-Lang plus Bogomolov type result for curves in $\mathbb{G}_\mathrm{m}^2$

Corentin PONTREAU

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Abstract

We prove a sharper so-called Mordell-Lang plus Bogomolov type result for curves lying in the two-dimensional linear torus. We mainly follow the approach of G. Rémond in [15], using Vojta and Mumford type inequalities. In the special case we consider, we improve Rémond's main result using a better Bogomolov property and an elementary arithmetic Bézout theorem.

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1 Introduction

Let X be a subvariety[1] of a torus[2] $\mathbb{G}_\mathrm{m}^g$ embedded in some projective space. The former Bogomolov's conjecture states that there exists an $\varepsilon > 0$ such that, provided that X is not a torsion subvariety, the set $X(\varepsilon)$ of points in $X(\overline{\mathbb{Q}})$ with height at most ε is not Zariski dense in X.

S. Zhang established the existence of such an ε in [17, Theorem 6.2], with possible dependence on the height of X (see [18, Theorem 1.10]). Be that as it may the infimum of the set of ε verifying this property has to depend on arithmetic properties of X, as we can see in the example of a torus coset (*i.e.*, a translate $\boldsymbol{\alpha} \cdot H$ of an algebraic subgroup H of $\mathbb{G}_\mathrm{m}^g$). However, if X is not a torus coset, one can find such an ε depending only on geometric invariants. Indeed, E. Bombieri and U. Zannier obtained in [4] a lower bound depending only on the ambient dimension g of the torus and on the degree of X (more precisely on an upper bound depending on the degree of a set of polynomial equations defining X). Bombieri and Zannier did not give an explicit expression for their bound. Later, by another approach, W. M. Schmidt [16, Theorem 4] obtained a completely explicit lower bound which is pluri-exponential in all parameters.

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[1] That is, a Zariski closed subset defined over a number field and geometrically irreducible.

[2] Most of the results mentioned in our historical overview have been generalized to the more general case of semi-abelian varieties.

In this paper, we shall use the embedding $\varphi : \mathbb{G}_{\mathrm{m}}^g \hookrightarrow \mathbb{P}^{2^g-1}$ defined as the composition

$$\mathbb{G}_{\mathrm{m}}^g \hookrightarrow (\mathbb{P}^1)^g \xhookrightarrow{\text{Segre}} \mathbb{P}^{2^g-1}.$$

Furthermore, for a subvariety X of $\mathbb{G}_{\mathrm{m}}^g$, we define $\deg X$ to be the degree of the Zariski closure of $\varphi(X)$ in $\mathbb{P}^{2^g-1}$. By the height $|\mathbf{x}|$ of a point $\mathbf{x} \in \mathbb{G}_{\mathrm{m}}^g(\overline{\mathbb{Q}})$, we shall mean the logarithmic Weil height of its image under φ in $\mathbb{P}^{2^g-1}$ (see Section 2 for more details).

More precisely S. David and P. Philippon proved that for $\varepsilon \approx (\deg X)^{-3\cdot 7^{\dim X}}$, the set $X(\varepsilon)$ is contained in a union of finitely many proper torus cosets contained in X (see [6, 7]). Moreover they estimated from above the sum of the degrees of the torus cosets by a quantity which is of the order of ε^{-1}. More recently F. Amoroso and S. David gave in [2] a sharper version where ε is approximately equal to $(\deg X)^{-1}$—in fact something essentially optimal—but without an effective estimate for the number of torus cosets.

The former Mordell-Lang's conjecture in the case of tori states that for any subgroup Γ of $\mathbb{G}_{\mathrm{m}}^g(\overline{\mathbb{Q}})$ of finite rank, the intersection $\Gamma \cap X(\overline{\mathbb{Q}})$ is contained in a union of finitely many proper torus cosets. This has been proved by M. Laurent (see [10]).

One can in fact give a more general statement, of the form "Mordell-Lang plus Bogomolov" (using the terminology of B. Poonen), by taking a thickening Γ_ε of Γ:

$$\Gamma_\varepsilon := \left\{ \mathbf{xy} \ : \ \mathbf{x} \in \Gamma, \ \mathbf{y} \in \mathbb{G}_{\mathrm{m}}^g(\overline{\mathbb{Q}}), \ |\mathbf{y}| \leq \varepsilon \right\}.$$

This corresponds to the set of points which are "close" to Γ with respect to the height. G. Rémond proved in [15] the following result:

Theorem A. *Let X be a subvariety of $\mathbb{G}_{\mathrm{m}}^g$ and Γ a subgroup of $\mathbb{G}_{\mathrm{m}}^g(\overline{\mathbb{Q}})$ of finite rank r. There exist torus cosets $B_1, \dots, B_S \subseteq X$ such that for all $\varepsilon \leq \varepsilon_0$*

$$X(\overline{\mathbb{Q}}) \cap \Gamma_\varepsilon \subseteq B_1 \cup \cdots \cup B_S \ ,$$

where $\varepsilon_0 = (\deg X)^{-g^2 m^{3m}}$, $S = (\deg X)^{(r+1)g^2 m^{3m^2}}$ and $m = \dim X + 1$.

Note that if Γ is the trivial group, Theorem A corresponds to the problem of Bogomolov, and if $\varepsilon = 0$ to the conjecture of Mordell-Lang. This has first been proved in a non-effective way by B. Poonen in [13], and afterwards by J.-H. Evertse (see [8]) but with pluri-exponential and polynomial values for ε_0^{-1} and S.

Specializing the proof of G. Rémond to curves, one can obtain that for any curve which is not a torus coset

$$\operatorname{Card} X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \leq \left(2^{28g+325} (\deg X)^{40} \right)^{r+1} ,$$

where $\varepsilon_0^{-1} = 2^{7g+183} (\deg X)^{28/3}$. In that case he proved in fact a more general result. First, define the notion of *truncated cone* around Γ by putting

$$\mathcal{C}(\Gamma, \varepsilon) := \left\{ \mathbf{xy} \ : \ \mathbf{x} \in \Gamma, \ \mathbf{y} \in \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}}), \ |\mathbf{y}| \leq \varepsilon(1 + |\mathbf{x}|) \right\}.$$

In the sequel we use the projective height $h(X)$ (which depends on the considered embedding φ) of X and the height $h_1(f)$ of a polynomial f, those are defined in Section 2. As we will see, in the case where X is a curve of $\mathbb{G}_{\mathrm{m}}^2$ and $f(x,y)=0$ an equation of X, these two quantities are comparable.

Theorem B (Rémond [15, Théorème 1.2])**.** *Let X be a curve of $\mathbb{G}_{\mathrm{m}}^g$ which is not a torus coset and Γ a subgroup of $\mathbb{G}_{\mathrm{m}}^g(\overline{\mathbb{Q}})$ of finite rank r. Then we have*

$$\operatorname{Card} X(\overline{\mathbb{Q}}) \cap \mathcal{C}(\Gamma, \varepsilon_1) \leq \left(2^{12g+229} (\deg X)^{33} \max\{1, h(X)\}\right)^{r+1} ,$$

where $\varepsilon_1^{-1} = 2^{12g+229} (\deg X)^{33} \max\{1, h(X)\}$.

Our first result is stated as follows (with the same embedding):

Theorem 1.1. *Let X be a curve of $\mathbb{G}_{\mathrm{m}}^2$ and Γ a subgroup of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of finite rank r. Assume that X is not a torus coset, then we have*

$$\operatorname{Card} X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \leq 2^{52r+51} (\deg X)^{3r+3} (\log \deg X)^{5r+6} ,$$

where $\varepsilon_0^{-1} := 2^{57} (\deg X)(\log \deg X)^5$.

Assume $f(x,y)=0$ is an equation of X. Since $\deg X = \deg_\varphi(X)$ is the sum of the partial degrees of f (see remark preceding Section 3) and since X is not a torus coset, we have $\deg X \geq 2$. Note that we improve here not only the bound for the number of points, but also the quantity ε_0. Moreover this quantity is essentially optimal, since one cannot expect, even when Γ is trivial, something greater than or equal to $c \cdot (\deg X)^{-1}$ for some positive constant $c > 0$.

In order to count in $X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0}$ the number of points with not too large height, we use a refined Bogomolov property ([12, Proposition 5.1]) for certain translates V of X, that is, an upper bound for the number of points with small height in V lying outside every non-trivial torus coset contained in V. This gives the main improvement in Theorem 1.1. To prove our refined Bogomolov property in [12], we first contruct *via* a Siegel lemma a polynomial vanishing on $V(2^{-10}\varepsilon_0)$ (the set of points of V with height lower or equal than $2^{-10}\varepsilon_0$). Using our control over the degree of this auxiliary polynomial and the fact that V is a hypersurface, we obtain *via* a zero lemma a divisor of V containing $V(2^{-10}\varepsilon_0)$. Then the geometric Bézout theorem gives an upper bound for the degree of this divisor, thus an upper bound for the number of points with small height when V is a curve (in other words when X is a curve). This explains why the result is only available for a curve lying in $\mathbb{G}_{\mathrm{m}}^2$.

Using the same argument as in the proof of Theorem B, we obtain the following result:

Theorem 1.2. *Let X be a curve of $\mathbb{G}_{\mathrm{m}}^2$ and Γ a subgroup of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of finite rank r. Assume that X is not a torus coset, then we have*

$$\operatorname{Card} X(\overline{\mathbb{Q}}) \cap \mathcal{C}(\Gamma, \varepsilon_1) \leq 2^{52r+51} (\deg X)^{2r+2} (\log \deg X)^{5r+6} \max\{(\deg X)^2, h_1(f)^r\} ,$$

where $\varepsilon_1^{-1} := 2^{51} (\deg X)^2 (\log \deg X)^5 \max\{1, h_1(f)\}$ and $f(x,y)=0$ is an equation of X.

We remark that, as in Theorem B, the bounds depend on the height of X (more precisely here on the height of f). The reason is that the set $\mathcal{C}(\Gamma, \varepsilon_1)$ is not invariant by translation by a point of Γ, contrary to Γ_{ε_1}.

Our Bogomolov property (Proposition 3.3) states that if a curve X is not a torus coset, then there are at most q_2 points in $X(\overline{\mathbb{Q}})$ of height lower than q_1^{-1}, for some quantities q_1, q_2 depending on the degree of X. With such a result one can obtain, *via* an argument of Schmidt (see [16, Theorem 5]) slightly modified, the following bound. Under the assumptions of Theorem 1.1, given $C \geq 1$ and $0 \leq \varepsilon \leq \frac{1}{4} q_1^{-1}$, there are at most

$$q_2 \cdot (4C \cdot q_1 + 1)^r \tag{1}$$

points in $\Gamma_\varepsilon \cap X$ with height smaller than C. In the final counting of Theorem 1.1, the main term comes from (1), choosing $C \approx (\deg X)^2$. In order to count the points of "large" height, we use a classical Vojta type inequality and a Mumford type gap principle.

Our paper follows mainly [15]. We first recall some definitions and facts about heights in Section 2. In Section 3, we give an elementary arithmetic Bézout theorem an recall the Bogomolov property of [12].

In Section 4, we recall G. Rémond's version of Vojta's inequality in the case of curves and give an optimized version of Mumford inequality. Finally, we prove the main theorems by a counting of the small and the large points.

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2 Heights

We will use several notions of height. Given a number field k we write $\mathcal{M}_k$ for the set of its places. For $v \in \mathcal{M}_k$, we write $|\cdot|_v$ for the absolute value having $|p|_v = p^{-1}$ if v lies above a prime number p, and requiring that the restriction of $|\cdot|_v$ to $\mathbb{Q}$ is the ordinary absolute value. We further denote by k_v the completion of k at v.

Let $\mathbf{a}$ be a point of $\mathbb{P}^n(k)$. We denote by h_1, h_2 and h_∞ the quantity

$$\sum_{v \in \mathcal{M}_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \log \|\mathbf{a}\|_v$$

where $\|\mathbf{a}\|_v$ is respectively the 1-norm, the 2 (euclidean) norm and the infinite (sup) norm associated to the absolute value $|\cdot|_v$ if v is archimedean:

$$|a_0|_v + \cdots + |a_n|_v, \quad \left(|a_0|_v^2 + \cdots + |a_n|_v^2\right)^{1/2}, \quad \max\{|a_0|_v, \ldots, |a_n|_v\}\ ,$$

and the infinite norm if v is non-archimedean. In particular $h_\infty(\mathbf{a})$ is the logarithmic Weil height of $\mathbf{a}$. Note that we have the following inequalities

$$h_2(\mathbf{a}) - \frac{1}{2}\log(n+1) \leq h_\infty(\mathbf{a}) \leq h_2(\mathbf{a}) \leq h_1(\mathbf{a}). \tag{2}$$

Furthermore, if B is a finite subset of $\overline{\mathbb{Q}}$ (like the set of coefficients of a polynomial), we will denote by $h_1(B), h_2(B)$ and $h_\infty(B)$ the height of the projective point defined by B.

For any $\mathbf{x} = (x_1, x_2) \in \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$, as in [15], we define $|\mathbf{x}|$ as the quantity

$$|\mathbf{x}| := h_\infty(1 : x_1 : x_2 : x_1x_2) = h_\infty(1 : x_1) + h_\infty(1 : x_2).$$

In particular we have, for all $\lambda \in \mathbb{Q}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$,

$$|\mathbf{x}\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}| \quad \text{and} \quad |\mathbf{x}^\lambda| = |\lambda||\mathbf{x}|.$$

Here $\mathbf{x}^\lambda$ is defined up to multiplication with a torsion point but for the value of $|\mathbf{x}^\lambda|$ this does not matter. We extend this height $|\cdot|$ to a norm on the $\mathbb{R}$-vector space $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{R}$ (the argument that $|\cdot|$ is a norm is due to Cassels, see [3] p. 137 for details). Addition and scalar multiplication on this vector space will be denoted respectively multiplicatively and exponentially, to be consistent with the operations on $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$. The "*angle*" between $\mathbf{x}$ and $\mathbf{y}$ is defined by:

$$\widehat{(\mathbf{x}, \mathbf{y})} := |\mathbf{y}^{\frac{1}{|\mathbf{y}|}} \cdot \mathbf{x}^{\frac{-1}{|\mathbf{x}|}}|.$$

The projective height $h(Z)$ of a subvariety Z of $\mathbb{G}_{\mathrm{m}}^2$ is defined by means of a Chow form of the Zariski closure of $\varphi(Z)$ in $\mathbb{P}^3$ (see [11]). Let us now consider the case of a hypersurface Z (a curve here) defined by some equation $g(x,y) = 0$. We recall that, by definition, the Mahler measure of g is the quantity

$$M(g) := \exp\left(\int_0^1\int_0^1 \log|g(e^{2i\pi u}, e^{2i\pi v})|\, du\, dv\right).$$

The Gauss-Mahler height $h_{GM}(g)$ of g (notation of [6]) is

$$h_{GM}(g) := \sum_{v \in \mathcal{M}_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \log M_v(g) \tag{3}$$

where $M_v(g)$ is the Mahler measure of $\sigma(g)$ if v is the archimedean place associated to σ and the Gauss norm (*i.e.* maximum norm) of g if v is finite. We have the following inequalities:

$$\begin{aligned} |h(Z) - 2h_1(g)| &\leq 9(\log 2)\cdot \deg Z \\ h_{GM}(g) \leq h_1(g) &\leq h_{GM}(g) + (\log 2)\cdot \deg Z. \end{aligned} \tag{4}$$

Indeed, on the one hand we know (see [6, Proposition 2.1 (v) and (vii)]) that

$$|2h_{GM}(g) - h(Z)| \leq 7(\log 2)\cdot \deg Z.$$

On the other hand, at infinite places, we know that the Mahler measure is smaller than the 1-norm and $\|g\|_1 \leq 2^{d_{x,g}+d_{y,g}} M(g)$ (see for instance [9, Lemma B.7.3.2]). Thus, using the equality $\deg Z = \deg_\varphi Z = d_{x,g} + d_{y,g}$, the sum of the partial degrees of g (see [14, p. 103]), we obtain (4).

3 Auxiliary results

We will need an arithmetic Bézout result, like [11, Théorème 3], for curves. But in this case one can give a slightly more precise statement with an elementary proof.

Lemma 3.1. [arithmetic Bézout for curves] *Let k be a number field and let $f, g \in k[x, y]$ be coprime polynomials. For any common zero $\mathbf{x}_0 = (x_0, y_0)$ of f and g one has*

$$|\mathbf{x}_0| \leq \deg_s(g) \cdot h_{GM}(f) + \deg_s(f) \cdot h_1(g) \ ,$$

where $\deg_s$ *is the sum of the partial degrees.*

Proof. By hypothesis the resultant $R := \mathrm{Res}_y(f, g)$ is non zero. Now since x_0 is a root of R, its height is bounded above by $h_{GM}(R)$ (see (3)):

$$h_\infty(1, x_0) \leq h_{GM}(R). \tag{5}$$

Denoting by $d_{x,f}, d_{y,f}, d_{x,g}, d_{y,g}$ the partial degrees of f and g with respect to the variables x and y, we can write f and g as

$$f(x, y) = \sum_{i=0}^{d_{y,f}} a_i(x) y^i \quad \text{and} \quad g(x, y) = \sum_{i=0}^{d_{y,g}} b_i(x) y^i.$$

Then we have

$$R(x) = \det \begin{pmatrix} a_0(x) & \dots & a_{d_{y,f}}(x) & & \\ & \ddots & & \ddots & \\ & & a_0(x) & \dots & a_{d_{y,f}}(x) \\ b_0(x) & \dots & b_{d_{y,g}}(x) & & \\ & \ddots & & \ddots & \\ & & b_0(x) & \dots & b_{d_{y,g}}(x) \end{pmatrix}$$

Let v be a finite place of k. We know from [1, Lemme 4.1.7, Corollaire 4.1.12] the following inequality:

$$M_v(R) \leq \sup_{t \in \mathbb{C}_v, |t|_v = 1} |R(t)|_v. \tag{6}$$

So let $t \in \mathbb{C}_v$ of absolute value 1. Expanding the determinant over all permutations we obtain, by (6) and the ultrametric inequality,

$$M_v(R) \leq \|f\|_v^{d_{y,g}} \cdot \|g\|_v^{d_{y,f}} \ , \tag{7}$$

where $\| \cdot \|_v$ is the Gauss norm.

Let us now consider archimedean places. Let $t \in \mathbb{C}$ of absolute value 1. We have:

$$\begin{aligned} |\mathrm{Res}(f(t,y),g(t,y))| = |R(t)| &= |a_{d_{y,f}}(t)|^{d_{y,g}} \cdot \prod_{\substack{a\in\mathbb{C}\\ f(t,a)=0}} |g(t,a)| \\ &\leq |a_{d_{y,f}}(t)|^{d_{y,g}} \cdot \prod_{\substack{a\in\mathbb{C}\\ f(t,a)=0}} \|g\|_1 \cdot \max\{1,|a|\}^{d_{y,g}} \\ &\leq M\left(f(t,y)\right)^{d_{y,g}} \cdot \|g\|_1^{d_{y,f}}. \end{aligned}$$

The integral over the unit circle of the logarithm gives:

$$\log M(R) \leq d_{y,g} \log M(f) + d_{y,f} \log \|g\|_1. \tag{8}$$

Now summing (7) and (8) over all places of k and using (5), we obtain

$$h_\infty(1,x_0) \leq d_{y,g} \cdot h_{GM}(f) + d_{y,f} \cdot h_1(g).$$

Similarly we have

$$h_\infty(1,y_0) \leq d_{x,g} \cdot h_{GM}(f) + d_{x,f} \cdot h_1(g).$$

Then the inequality of the lemma follows. □

Corollary 3.2. *Let X be a curve of $\mathbb{G}_\mathrm{m}^2$ and $\boldsymbol{\alpha} \in \mathbb{G}_\mathrm{m}^2(\overline{\mathbb{Q}})$ which does not belong to the stabilizer*[3] *of X. Then for every $\mathbf{x}$ in $X \cap \boldsymbol{\alpha}^{-1} \cdot X$ we have:*

$$|\mathbf{x}| \leq 2h_1(f)(\deg X) + |\boldsymbol{\alpha}|(\deg X)^2.$$

Proof. Assume X is defined by some equation $f(x,y) = 0$ where $f(x,y) := \sum_{i\leq d_{x,f}, j\leq d_{y,f}} a_{i,j}x^iy^j$. Let $(\alpha,\beta) := \boldsymbol{\alpha}$ and $g(x,y) := f(\alpha \cdot x, \beta \cdot y)$. Then $g(x,y) = 0$ is an equation of $\boldsymbol{\alpha}^{-1} \cdot X$. Note that f and g are coprime since $\boldsymbol{\alpha}$ does not belong to the stabilizer of X. Moreover we have $d_{x,f} + d_{y,f} = \deg_s(f) = \deg X$ and similarly for g. Hence from Lemma 3.1 and (4) we get

$$\begin{aligned} |\mathbf{x}_i| &\leq \deg X \cdot (h_{GM}(f) + h_1(g)) \\ &\leq \deg X \cdot \Big(2h_1(f) + d_{x,f}h_\infty(1,\alpha) + d_{y,f}h_\infty(1,\beta)\Big) \\ &\leq \deg X \cdot \Big(2h_1(f) + \deg X |\boldsymbol{\alpha}|\Big). \end{aligned}$$

□

Instead of the Bogomolov type inequality from [7], we use the following one from [12] (see Proposition 5.1), giving a better upper bound for the number of "small" points in a curve:

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[3] That is, the set $\{\mathbf{x} \in \mathbb{G}_\mathrm{m}^2(\overline{\mathbb{Q}}) \ : \ \mathbf{x} \cdot X = X\}$.

Proposition 3.3. *Let Z be a curve of $\mathbb{G}_{\mathrm{m}}^2$. If Z is not a torus coset then*

$$\operatorname{Card}\left\{\mathbf{z} \in Z(\overline{\mathbb{Q}}) \ : \ |\mathbf{z}| \leq q_1^{-1}\right\} \leq q_2$$

where $q_1 := 2^{47}(\deg Z)(\log \deg Z)^5$ *and* $q_2 := 2^{50}(\deg Z)^2(\log \deg Z)^6$.

Remark: In [12, Proposition 5.1] we used instead of q_1 and q_2 the following quantities:

$$\tilde{q}_1 := 10^{11}\omega(\log \omega')^5 \quad \text{and} \quad \tilde{q}_2 := 2 \cdot 10^{11}\omega^2(\log \omega')^6$$

where $\omega := \deg_\iota(Z)$ is the degree of the Zariski closure of the image of Z by $\iota : \mathbb{G}_{\mathrm{m}}^2 \hookrightarrow \mathbb{P}^2$, $(x, y) \mapsto (1 : x : y)$ and where $\omega' := \max\{16, \deg_\iota(Z)\}$. But we have $\deg_\iota(Z) \leq (\deg Z) \leq 2\deg_\iota(Z)$ and $h_\infty(\iota(\mathbf{z})) \leq |\mathbf{z}| \leq 2h_\infty(\iota(\mathbf{z}))$. Moreover since $\deg Z \geq 2$ we also have $4 \log \deg Z \geq \log \omega'$. Thus the proposition is valid.

Suppose Z is given by the equation $g(x, y) = 0$. Then one has

$$2h_{GM}(g) \geq q_1^{-1} \cdot \deg Z. \tag{9}$$

This is a consequence of a theorem of S. Zhang. Indeed, for any real number θ and any variety V, let us denote $V(\theta)$ the subset of points $\boldsymbol{\alpha}$ in V such that $|\boldsymbol{\alpha}| \leq \theta$. The *essential minimum* of V is defined as follow

$$\hat{\mu}_{\text{ess}}(V) := \inf\left\{\theta > 0 \mid \overline{V(\theta)}^{\text{Zar}} = V\right\} \ ,$$

where $\overline{V(\theta)}^{\text{Zar}}$ is the Zariski closure of $V(\theta)$. Thus from Proposition 3.3 it follows $\hat{\mu}_{\text{ess}}(Z) \geq q^{-1}$. A particular case of a result of S. Zhang (see [17, theorem 5.2] and [18, theorem 1.10][4]), gives the following inequality

$$\hat{h}(V) \geq \hat{\mu}_{\text{ess}}(V) \cdot \deg V.$$

Moreover by [6, Proposition 2.1 (vii)] we have $2h_{GM}(g) = \hat{h}(Z)$, so (9) is proved.

The following lemma of G. Rémond (see [15, Lemme 2.1]) generalizes an argument of Schmidt (see [16, Theorem 5]).

Lemma 3.4. *Let Γ a subgroup of $\mathbb{G}_{\mathrm{m}}^2(\overline{\mathbb{Q}})$ of rank r and $\varepsilon \geq 0$*

1. *Let ρ and μ be two real numbers such that $\rho \geq 0$, $\mu > 0$ and $\varepsilon \leq \rho/(2\mu)$. Then there exists a subset E of Γ such that* $\operatorname{Card} E \leq (4\mu + 3)^r$ *and*

$$\{x \in \Gamma_\varepsilon \ : \ |\mathbf{x}| \leq \rho\} \subseteq \bigcup_{\mathbf{y} \in E} \{x \in \Gamma_\varepsilon \ : \ |\mathbf{x}\mathbf{y}^{-1}| \leq \rho/\mu\}.$$

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[4] In [5, Corollaire 3.2], there is also a simpler proof in the case of abelian varieties, directly adaptable to $\mathbb{G}_{\mathrm{m}}^g$.

2. *Let* c_1 *and* c *be two positive numbers*[5] *such that* $c_1 \geq 1$ *and* $\varepsilon \leq c/(8c_1)$. *Then there exists a partition of* $\{\mathbf{x} \in \Gamma_\varepsilon \ : \ |\mathbf{x}| \geq c\}$ *made of at most* $(1+8c_1)^r$ *sectors in which any two points* $\mathbf{x}, \mathbf{y}$ *verify* $\widehat{(\mathbf{x},\mathbf{y})} \leq c_1^{-1}$.

3. *If* $c \geq 8$ *and* $\varepsilon \leq (10c_1)^{-1}$ *then we can replace* Γ_ε *by* $\mathcal{C}(\Gamma, \varepsilon)$ *in the previous statement.*

4 Inequalities of Vojta and Mumford

In what follows, we consider a geometrically irreducible curve X of $\mathbb{G}_{\mathrm{m}}^2$ defined over $\overline{\mathbb{Q}}$ which is not a torus coset. We introduce the following quantities:

$$\begin{cases} \Lambda & := & \left(3 \cdot 2^{10} (\deg X)^5\right)^{1/3} \leq (\deg X)^6 \qquad \text{since} \quad \deg X \geq 2 \\ c_1 & := & 2^8 (\deg X)^2 \\ c_2 & := & \Lambda^6 \\ c_3 & := & \Lambda^{14} \max\{h(X), 1\} \end{cases}$$

First we recall a particular case of a result of G. Rémond ([15, Théorème 3.1].

Theorem C (Vojta's inequality)**.** *For any* $\mathbf{x}_1, \mathbf{x}_2 \in X(\overline{\mathbb{Q}})$ *such that*

$$|\mathbf{x}_1| \geq c_3 \quad \textit{and} \quad \widehat{(\mathbf{x}_1,\mathbf{x}_2)} \leq c_1^{-1}$$

we have $|\mathbf{x}_2| < c_2 |\mathbf{x}_1|$.

Using Corollary 3.2 we get the following improved version of [15, proposition 4.2] in the case of curves in $\mathbb{G}_{\mathrm{m}}^2$.

Proposition 4.1 (Mumford's inequality)**.** *Let* $c_4 := 4 \deg X \cdot h_1(f)$ *and* $\mathbf{x} \in X(\overline{\mathbb{Q}})$ *such that* $|\mathbf{x}| > c_4$. *There are at most* $(\deg X)^2$ *points* $\mathbf{y}$ *in* $X(\overline{\mathbb{Q}})$ *such that*

$$\widehat{(\mathbf{x},\mathbf{y})} \leq \frac{1}{c_1} \quad \textit{and} \quad \Big| |\mathbf{x}| - |\mathbf{y}| \Big| \leq \frac{1}{c_5} |\mathbf{x}| \tag{10}$$

where $c_5 := 4(\deg X)^2$.

Proof. Consider $\mathbf{y} \in X$ satisfying (10) and let $\boldsymbol{\alpha} := \mathbf{x}\mathbf{y}^{-1}$. One has

$$\begin{aligned} |\boldsymbol{\alpha}| &= |\mathbf{x}| \cdot \left| \mathbf{y}^{1/|\mathbf{x}|} \cdot \mathbf{x}^{-1/|\mathbf{x}|} \right| \\ &= |\mathbf{x}| \cdot \left| \mathbf{y}^{1/|\mathbf{y}|} \cdot \mathbf{x}^{-1/|\mathbf{x}|} \cdot \mathbf{y}^{1/|\mathbf{x}| - 1/|\mathbf{y}|} \right| \\ &\leq |\mathbf{x}| \cdot \left(\left| \mathbf{y}^{1/|\mathbf{y}|} \cdot \mathbf{x}^{-1/|\mathbf{x}|} \right| + \left| \frac{1}{|\mathbf{x}|} - \frac{1}{|\mathbf{y}|} \right| \cdot |\mathbf{y}| \right) \\ &< \frac{|\mathbf{x}|}{2(\deg X)^2}. \end{aligned}$$

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[5] In [15, Lemme 2.1] the author assume $c \geq 1$, but the hypothesis c positive is sufficient.

Thus,

$$2h_1(f)\cdot\deg X+|\boldsymbol{\alpha}|(\deg X)^2<2h_1(f)\cdot\deg X+\frac{|\mathbf{x}|}{2}\le|\mathbf{x}|$$

From Corollary 3.2 we know that $\boldsymbol{\alpha}:=\mathbf{x}^{-1}\cdot\mathbf{y}$ must belong to the stabilizer of X. On the other hand the stabilizer of X is an algebraic subgroup equal to $\cap_{\mathbf{z}\in X}\mathbf{z}^{-1}\cdot X$. Since X is not a torus coset, the dimension of this intersection is zero. Thus, by the geometric Bézout theorem, its cardinality is at most $(\deg X)^2$. □

Remark: We have

$$\frac{c_3}{c_4}\le\Lambda^{14}. \tag{11}$$

Indeed, using (4) and $\deg X\ge 2$ we get

$$\begin{aligned}\frac{c_3}{c_4} &= \frac{\Lambda^{14}\cdot\max\{1,h(X)\}}{4(\deg X)\cdot h_1(f)}\\ &\le \frac{\Lambda^{14}\cdot(2h_1(f)+9(\log 2)\deg X)}{4(\deg X)\cdot h_1(f)}\\ &\le \frac{\Lambda^{14}}{2}+\frac{9\log 2}{4\cdot h_1(f)}.\end{aligned}$$

Now we know from (4) and (9) that

$$2h_1(f)\ge 2h_{GM}(f)\ge q_1^{-1}\cdot\deg X. \tag{12}$$

Now since $\Lambda=\left(3\cdot 2^{10}(\deg X)^5\right)^{1/3}$ and $q_1=2^{47}(\deg X)(\log\deg X)^5$ we obtain

$$\frac{9\log 2}{4\cdot h_1(f)}\le\frac{1}{2}\Lambda^{14}.$$

Corollary 4.2. *Let* $N:=2^{20r+10}\cdot(\deg X)^{2r+4}\cdot\log(\deg X)$. *Then*

$$\operatorname{Card}\left\{\mathbf{x}\in X(\overline{\mathbb{Q}})\cap\Gamma_{\varepsilon_0}\ :\ |\mathbf{x}|>c_4\right\}\le N$$

where $\varepsilon_0:=2^{-10}q_1^{-1}$.

Proof. It follows from (12) that $\varepsilon_0:=\frac{q_1^{-1}}{2^{10}}\le\frac{h_1(f)}{2^9\deg X}=\frac{c_4}{8c_1}$. Consequently we can apply the second assertion of Lemma 3.4 with $c=c_4$. Thus we obtain a partition into at most $(1+8c_1)^r$ sectors such that if $\mathbf{x},\mathbf{y}$ are points in the same sector, then $\widehat{(\mathbf{x},\mathbf{y})}\le\frac{1}{c_1}$. Consider one of these sectors. First consider the set of points $\mathbf{x}$ in this sector satisfying $c_4<|\mathbf{x}|<c_3$. This set can be divided into at most $\frac{\log(c_3/c_4)}{\log(1+1/c_5)}+1$ subsets such that any two points $\mathbf{x},\mathbf{y}$ in the same such subset verify $||\mathbf{x}|-|\mathbf{y}||<1/c_5$. According to Proposition 4.1, each of the subsets has cardinality at most $(\deg X)^2$. So, using (11), it follows that the set of points $\mathbf{x}$ in the sector with $c_4<|\mathbf{x}|<c_3$ has cardinality at most

$$\left(\frac{\log(c_3/c_4)}{\log(1+1/c_5)}+1\right)\cdot(\deg X)^2\le\frac{14\log\Lambda}{\log(1+1/c_5)}(\deg X)^2. \tag{13}$$

Now consider the elements $\mathbf{x}$ such that $|\mathbf{x}| \geq c_3$ and let us order those points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ such that $|\mathbf{x}_0| \leq |\mathbf{x}_1| \leq |\mathbf{x}_2| \leq \ldots$. Again by Proposition 4.1 we have for any $i, j \geq 0$

$$|\mathbf{x}_{i\cdot(\deg X)^2+j}| \geq (1+1/c_5)^i |\mathbf{x}_j|$$

So if $i \geq \log(c_2)/\log(1+1/c_5)$ then

$$|\mathbf{x}_{i\cdot(\deg X)^2}| \geq c_2 |\mathbf{x}_0|$$

which is impossible, by Vojta's inequality (Theorem C). We conclude that there are at most

$$\left(\frac{\log(c_2)}{\log(1+1/c_5)}+1\right)\cdot(\deg X)^2 \leq \left(\frac{4\log\Lambda}{\log(1+1/c_5)}+1\right)\cdot(\deg X)^2$$

points with $|\mathbf{x}| \geq c_3$ in the sector under consideration.

Combining this to (13), it follows that the set $\big\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \ : \ |\mathbf{x}| > c_4\big\}$ has cardinality at most

$$\left(\frac{18\log\Lambda}{\log(1+1/c_5)}+2\right)\cdot(\deg X)^2\cdot(1+8c_1)^r.$$

Using $\log(1+1/c_5) \geq (2c_5)^{-1}$ and $\Lambda \leq (\deg X)^6$, this bound can be estimated from above by

$$\big(18\times 6\log(\deg X)\cdot 2\times 4(\deg X)^2+2\big)\cdot(\deg X)^2\cdot\big(1+8(2^8\deg X)^2\big)^r,$$

which in turn is smaller than

$$2^{10}\cdot(\deg X)^4\log(\deg X)\cdot\big(2^{20}(\deg X)^2\big)^r = N.$$

□

5 Final counting

5.1 Proof of Theorem 1.1

For each $\mathbf{y}$ in Γ, let us consider an equation $f_{\mathbf{y}}(x,y)=0$ of $\mathbf{y}^{-1}\cdot X$.

First assume that there exists $\mathbf{y}_0$ in Γ such that $h_1(f_{\mathbf{y}_0}) \leq 2(\log 2)\cdot(\deg X)$. Since Γ_{ε_0} is invariant by multiplication by an element of Γ, the cardinality of $\mathbf{y}^{-1}\cdot X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0}$ does not depend on $\mathbf{y} \in \Gamma$. So there is no loss of generality to assume that $f_{\mathbf{y}_0} = f_{(1,1)} = f$.

We now apply the first assertion of Lemma 3.4 with $\rho = c_4$ and $\mu = c_4 q_1$. Since $\varepsilon_0 \leq (2q_1)^{-1}$, one can cover the set $\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \ : \ |\mathbf{x}| \leq c_4\}$ by $(4c_4q_1+3)^r$ "semiballs" of radius q_1^{-1} centred at a point of Γ. By Proposition 3.3 in such a ball we have at most q_2 points. Using $c_4 = 4(\deg X)\cdot h_1(f) \leq 8(\log 2)\cdot(\deg X)^2$ we obtain

$$\begin{aligned}
\mathrm{Card}\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \ : \ |\mathbf{x}| \leq c_4\} &\leq q_2\cdot(4c_4q_1+3)^r \\
&\leq q_2\cdot\big(2^5(\deg X)^2\cdot q_1\big)^r \\
&\leq 2^{52r+50}(\deg X)^{3r+2}(\log\deg X)^{5r+6}
\end{aligned}$$

Assume now that for all $\mathbf{y} \in \Gamma$ we have $h_1(f_{\mathbf{y}}) > 2(\log 2) \cdot (\deg X)$. In particular by (4), one has $2h_{GM}(f_{\mathbf{y}}) > h_1(f_{\mathbf{y}})$. We may also assume that $h_1(f)$ is mainly the smallest among all the $f_{\mathbf{y}}$, more precisely that $\inf_{\mathbf{y}\in\Gamma} h_1(f_{\mathbf{y}}) \geq \frac{1}{2}h_1(f)$ (again because the cardinality of $\mathbf{y}^{-1} \cdot X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0}$ does not depend on $\mathbf{y} \in \Gamma$).

By [6, Proposition 5.4] we have[6], for all $\mathbf{y} \in \Gamma$

$$\operatorname{Card}\left\{\mathbf{z} \in \mathbf{y}^{-1}X(\overline{\mathbb{Q}}) \ : \ |\mathbf{z}| \leq \frac{1}{2^5e^2} \cdot \frac{2h_{GM}(f_{\mathbf{y}})}{\deg X}\right\} \leq q_2' \tag{14}$$

where $q_2' := 2^{58} \cdot (\deg X)^4(\log \deg X)^2$. Once more we apply the first assertion of Lemma 3.4 with $\rho = c_4$ and $\mu = c_4q_1'$, where

$${q_1'}^{-1} := \frac{1}{2^5e^2} \cdot \frac{2h_{GM}(f_{\mathbf{y}})}{\deg X}.$$

By the previous considerations and by (4) and (9) we have

$${q_1'}^{-1} \geq \frac{1}{2^5e^2} \cdot \frac{h_1(f)}{\deg X} \geq \frac{q_1^{-1}}{2^9}.$$

Thus $\varepsilon_0 = 2^{-57}(\deg X)^{-1}(\log \deg X)^{-2} = 2^{-10}q_1^{-1} \leq (2q_1')^{-1}$.

So we can cover the set $\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \ : \ |\mathbf{x}| \leq c_4\}$ by $(4c_4q_1' + 3)^r$ "semiballs" of radius ${q_1'}^{-1}$ centred at a point of Γ. By (14) in such a ball we have at most q_2' points. It follows

$$\begin{aligned}\operatorname{Card}\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} \ : \ |\mathbf{x}| \leq c_4\} &\leq q_2' \cdot (4c_4q_1' + 3)^r \\ &\leq q_2' \cdot \left(4 \cdot 6 \cdot 2^6e^2 \cdot (\deg X)^2 + 3\right)^r \\ &\leq 2^{14r+58} \cdot (\deg X)^{2r+4}(\log \deg X)^2\end{aligned}$$

In both cases, using Corollary 4.2 for the "large" points we finally get[7]

$$\begin{aligned}\operatorname{Card} X(\overline{\mathbb{Q}}) \cap \Gamma_{\varepsilon_0} &\leq N + 2^{52r+50}(\deg X)^{3r+3}(\log \deg X)^{5r+6} \\ &\leq 2^{52r+51} \cdot (\deg X)^{3r+3}(\log \deg X)^{5r+6}.\end{aligned}$$

Remark: We could improve the bound for the number of points in Theorem 1.1 if we had the Gauss-Mahler height instead of h_1 in Lemma 3.1. Indeed in this case it would be possible to replace $c_4 := 4 \deg X \cdot h_1(f)$ by $c_4 \approx \deg X \cdot h_{GM}(f)$ and, as in (14), it would give $2r + 4$ instead of $3r + 3$ in the exponent of $\deg X$ in our final bound.

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[6] we use also [6, Proposition 2.1 (vii)], which says that $\hat{h}(\mathbf{y}^{-1}X) = 2h_{GM}(f_{\mathbf{y}})$

[7] we also assume $r \geq 1$, since the case $r = 0$ is given by Proposition 3.3

5.2 Proof of Theorem 1.2

Let us denote $c := 4(\deg X) \cdot \max\{1, h_1(f)\}$ and $\varepsilon_1^{-1} := 4q_1 c$. Since $\deg X \geq 2$ we have $c \geq 8$ Moreover we have

$$\varepsilon_1 \leq \frac{1}{16q_1 \deg X} = \frac{1}{2^{51}(\deg X)^2(\log\deg X)^5} \leq \frac{1}{10 \times 2^8(\deg X)^2} = \frac{1}{10c_1}.$$

Then one can apply the same argument as in the proof of Corollary 4.2, applying point 3. of Lemma 3.4 instead of point 2. We thus obtain

$$\mathrm{Card}\left\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \mathcal{C}(\Gamma, \varepsilon_1) \ : \ |\mathbf{x}| > c\right\} \leq N.$$

To count the points of height smaller than c in $X(\overline{\mathbb{Q}}) \cap \mathcal{C}(\Gamma, \varepsilon_1)$, we boil down the problem to the intersection of X with $\Gamma_{\varepsilon'}$ where $\varepsilon' := (1+c)\frac{\varepsilon_1}{1-\varepsilon_1}$. More precisely we have

$$\{\mathbf{x} \in \mathcal{C}(\Gamma, \varepsilon_1) \ : \ |\mathbf{x}| \leq c\} \subseteq \{\mathbf{x} \in \Gamma_{\varepsilon'} \ : \ |\mathbf{x}| \leq c\}.$$

Indeed, if we consider an element of the first set $\mathbf{x} = \mathbf{y}\mathbf{z}$ with $\mathbf{y} \in \Gamma$ and $|\mathbf{z}| \leq \varepsilon_1(1 + |\mathbf{y}|) \leq \varepsilon_1(1 + |\mathbf{x}| + |\mathbf{z}|)$, then

$$(1-\varepsilon_1)|\mathbf{z}| \leq \varepsilon_1(1 + |\mathbf{x}|) \leq \varepsilon_1(1+c).$$

Now we notice that $\varepsilon' = (1+c)\frac{\varepsilon_1}{1-\varepsilon_1} \leq 2 \cdot \varepsilon_1 \cdot c = (2q_1)^{-1}$, so we can again apply point 1. of Lemma 3.4 with $\rho = c$ and $\mu = q_1 c$, together with Proposition 3.3. It follows

$$\mathrm{Card}\left\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \mathcal{C}(\Gamma, \varepsilon_1) \ : \ |\mathbf{x}| \leq c\right\} \leq q_2 \cdot (4 \cdot c \cdot q_1 + 3)^r.$$

Hence the set $\left\{\mathbf{x} \in X(\overline{\mathbb{Q}}) \cap \mathcal{C}(\Gamma, \varepsilon_1)\right\}$ has cardinality at most

$$\begin{aligned} & N + q_2 \cdot \left(2^5(\deg X)\max\{1, h_1(f)\} \cdot q_1\right)^r \\ \leq \quad & N + 2^{52r+50} \cdot (\deg X)^{2r+2} \cdot (\log\deg X)^{5r+6} \cdot \max\{1, h_1(f)\}^r \\ \leq \quad & 2^{52r+51} \cdot (\deg X)^{2r+2} \cdot (\log\deg X)^{5r+6} \cdot \max\{(\deg X)^2, h_1(f)^r\}. \end{aligned}$$

Note that if for all $\mathbf{y} \in \Gamma$ the polynomial $f_{\mathbf{y}}$ has height h_1 greater than $2(\log 2) \cdot \deg X$ then, as in Subsection 5.1, one can get rid of $h_1(f)$ in the final bound. But if one of the $f_{\mathbf{y}}$ has height h_1 lower than $2(\log 2) \cdot \deg X$, one cannot assume that $f_{\mathbf{y}} = f$, since $\mathcal{C}(\Gamma, \varepsilon_1)$ is not invariant by translation by a point of Γ, contrary to Γ_{ε_1}.

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