Geometric lower bounds for the normalized height of hypersurfaces

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We are here concerned in the Bogomolov's problem for the hypersurfaces; we give a geometric lower bound for the height of a hypersurface of $\mathbb{G}_m^n$ (*i.e.* without condition on the field of definition of the hypersurface) which is not a translate of an algebraic subgroup of $\mathbb{G}_m^n$. This is an analogue of a result of F. Amoroso and S. David who give a lower bound for the height of non-torsion hypersurfaces defined and irreducible over the rationals.

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1. Introduction

Let ε be a positive number. In 1933, D. H. Lehmer asked whether or not there exists a monic polynomial f with integer coefficients, such that the product of the absolute values of those roots of f which lie outside the unit circle, lies between 1 and $1+\varepsilon$. This product can be seen as the *height* of the algebraic subset of $\mathbb{G}_m$ defined by f, which describes in a certain sense the arithmetic complexity of this algebraic subset. In [9], P. Philippon defines, *via* the Chow forms, the notion of *normalized* height $\hat{h}$ for subvarieties of abelian varieties, then transpose the notion to subvarieties of $\mathbb{G}_m^n$ with S. David in [6]. In the case of a hypersurface V of $\mathbb{G}_m^n$ (embedded in the canonical way into $\mathbb{P}_n$), defined over $\mathbb{Q}$ by a polynomial F in $\mathbb{Z}[x_1,\dots,x_n]$, irreducible over $\mathbb{Z}$, the height of V is the logarithm of Mahler's measure of F:

$$\hat{h}(V) = \log M(F) := \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\left|F(e^{i\theta_1},\dots,e^{i\theta_n})\right| d\theta_1\dots d\theta_n.$$

More generally, if V is defined over a number field k, we have:

$$\hat{h}(V) = \frac{1}{[k:\mathbb{Q}]}\sum_{v\in\mathcal{M}_k}[k_v:\mathbb{Q}_v]\log M_v(F)\ ,$$

where $M_v(F)$ is Mahler's measure of $\sigma(F)$ if v is archimedean, associated to the embedding σ and $M_v(F) := \max\{|\operatorname{coeff}(F)|_v\}$ if v is ultrametric.

Moreover, if F is a non zero polynomial with coefficients in $\overline{\mathbb{Q}}$, we will denote by $h(F)$ the Weil height of the projective point defined by its coefficients (*c.f.* page 4).

The normalized height is always non-negative, moreover, for any subvariety W of $\mathbb{G}_m^n$, we know that $\hat{h}(W)$ is zero if and only if W is a union of torsion varieties (*i.e.* translate of subtori of $\mathbb{G}_m^n$ by torsion points): this result has been proved in particular by Lawton [8] in the case of hypersurfaces in $\mathbb{G}_m^n$ and by Zhang [12, theorem 6.2] in the general case.

Lehmer's conjecture gives a lower bound for Mahler's measure of a 1-variable polynomial with integer coefficients, which can be easily generalized to hypersurfaces (this is a particular case of the conjecture 1.4 of [2]):

Conjecture 1.1. *Let $n \geq 1$ be an integer. For any hypersurface V in $\mathbb{G}_m^n$ defined and irreducible over $\mathbb{Q}$ which is not a union of torsion subvarieties, we have:*

$$\hat{h}(V) \geq c_a \ ,$$

where $c_a > 0$ is an absolute constant.

Remark 1.2. To prove this conjecture, it is enough to find a lower bound for Mahler's measure for 1-variable polynomials. Indeed, one can write Mahler's measure of a n-variables polynomial as the limit of Mahler's measure of 1-variable polynomials. More precisely, for any $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n$ let's denote :

$$\mu(\mathbf{r}) := \min\left\{\|\mathbf{u}\|_\infty \mid \mathbf{u} \in \mathbb{Z}^n, \mathbf{u} \neq 0, < \mathbf{u}, \mathbf{r} >= 0\right\} \ ,$$

where $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$. For any $F \in \mathbb{Z}[x_1, \ldots, x_n]$, let us consider the 1-variable polynomial $F_{\mathbf{r}}(x) := F(x^{r_1}, \ldots, x^{r_n})$. Boyd proved the following result (see [4, lemma 2]) :

$$\lim_{\mu(\mathbf{r})\to\infty} \sup M(F_{\mathbf{r}}) \leq M(F).$$

Now we have to prove that if $M(F)$ is not equal to 1 so is $M(F_{\mathbf{r}})$ for infinitely many $\mathbf{r} \in \mathbb{Z}^n$. In such a case and if the conjecture is true for $n = 1$, we would have $M(F) \geq c_a$. For this we use a lemma of A. Schinzel (see [4, lemma 3]) which claims that if $M(F_{\mathbf{r}})$ is equal to 1, then there exists $\mathbf{u} \in \mathbb{Z}^n$ such that $< \mathbf{u}, \mathbf{r} >= 0$ and $\|\mathbf{u}\|_\infty \leq c(F)$, where $c(F)$ is a constant which only depends on F. In particular, $M(F_{\mathbf{r}})$ is equal to 1 for at most finitely many $\mathbf{r} \in \mathbb{Z}^n$.

2. Known results

In the direction of the conjecture 1.1, F. Amoroso and S. David (see [2, théorème 1.7]) obtain the following result (which is stated here in a slightly weaker form):

Theorem 2.1. *Let V be a hypersurface of $\mathbb{G}_m^n$ defined over $\mathbb{Q}$ and $\mathbb{Q}$-irreducible of degree ω. If V is not a union of translated algebraic subgroups by torsion points, then we have:*

$$\hat{h}(V) \geq \frac{c}{(n+1)^5} \, \frac{\big(\log\big((n+1)\log((n+1)\omega)\big)\big)^{2+1/(n-s)}}{\big(\log\big((n+1)\omega\big)\big)^{1+2/(n-s)}} \ ,$$

where c is an absolute constant and s is the dimension of the stabilizer of V.

We will now consider lower bounds for the height of varieties with no condition on their field of definition. A conjecture stated in [6, conjecture 1.1] suggests an analogue of the previous one for geometrically irreducible varieties which is, for hypersurface the following.

Conjecture 2.2. *Let V be a geometrically irreducible hypersurface of $\mathbb{G}_m^n$. If V is not a translated subtorus of $\mathbb{G}_m^n$, then we have:*

$$\hat{h}(V) \geq c_g \ ,$$

where $c_g > 0$ is an absolute constant.

Remark 2.3. Let us compare conjecture 1.1 and conjecture 2.2. If we first consider a variety $\boldsymbol{\alpha} \cdot H$, where H is an algebraic subtorus of $\mathbb{G}_m^n$ and $\boldsymbol{\alpha}$ a non torsion point, by conjecture 1.1 we have:

$$\hat{h}(\boldsymbol{\alpha} \cdot H) \geq \frac{c_a}{[\mathbb{Q}(\boldsymbol{\alpha}) : \mathbb{Q}]} \ ,$$

but conjecture 2.2 gives no information on $\hat{h}(\boldsymbol{\alpha} \cdot H)$.

On the other hand, let us consider in $\mathbb{G}_m^3$, for all integer r, the subvariety V_r defined by the polynomial $(x+y)^r - 5z^r$. By conjecture 1.1 we have:

$$\hat{h}(V_r) \geq c_a .$$

Moreover, by conjecture 2.2 we have:

$$\hat{h}(V_r) \geq r \cdot c_g \ ,$$

since $(x+y)^r - 5z^r = \prod_\zeta \big(x + y - 5^{1/r}\zeta \cdot z\big)$, where ζ runs over all the r^{th} roots of unity. So if r is large enough, conjecture 2.2 is better.

The case of a translated subtorus is essentially the Lehmer's problem for points; in general it is not possible to obtain a lower bound for the height of a subvariety V depending only on the geometric degree without any condition on V, we will come back to this topic in the next section.

As remarks the referee, using [6, Théorème 1.6] (reduction argument) and [3, théorème 1.5] (explicit lower bound for a curve), we obtain conjecture 2.2 up to an "ε". But since the considered embedding ($\mathbb{G}_m^n \hookrightarrow \mathbb{P}_1^n$) in [6, Théorème 1.6] is different, this gives a non absolute constant (about $\frac{2^{-70}}{n!}$, by comparison of the normalized heights, see [6, Proposition 2.1, (vii)]). However, the statement of [6, Théorème 1.6] still works for the embedding $\mathbb{G}_m^n \hookrightarrow \mathbb{P}_n$ we consider in this article. More precisely we have, for all geometrically irreducible hypersurface V of $\mathbb{G}_m^n$ which is not a translated subtorus:

$$\hat{h}(V) \geq 2^{-70} \frac{(\log\log(\omega+4))^4}{(\log\log(\omega+2))^5}. \tag{2.4}$$

We intend to prove a completely explicit analogue of the theorem 2.1 in the case of the geometrically irreducible hypersurfaces:

Theorem 2.4. *Let V be a geometrically irreducible hypersurface of $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$ of degree ω. If V is not a translated subtorus of $\mathbb{G}_m^n$, then we have:*

$$\hat{h}(V) \geq \frac{10^{-14}}{n^8} \frac{(\log(n\log\omega'))^{2+2/(n-s-1)}}{(\log\omega')^{1+4/(n-s-1)}} ,$$

where $\omega' := \max\{n\omega, 16\}$ and s is the dimension of the stabilizer of V.

In comparison to (2.4), this result is significant for varieties having trivial stabilizer (which is the "general" case) and ω large enough compared to n.

3. Auxiliary results

Let us consider the canonical embedding ι from $\mathbb{G}_m^n$ into $\mathbb{P}_n$. As for the hypersurfaces, the normalized height of 0-dimensional varieties -*i.e.* a point $\boldsymbol{\alpha}$- can be easily described. Indeed, if k is a number field containing its coordinates, one can show that the normalized height of $\boldsymbol{\alpha}$ is the Weil height:

$$\hat{h}(\{\boldsymbol{\alpha}\}) = h(\boldsymbol{\alpha}) := \frac{1}{[k:\mathbb{Q}]} \sum_{v\in\mathcal{M}_k} [k_v:\mathbb{Q}_v] \log\max\{1, |\alpha_1|_v, \dots, |\alpha_n|_v\}.$$

Before coming back to the height of a translated subtorus, let's introduce another quantity. For any real number θ and any variety V, let's denote $V(\theta)$ the subset of points $\boldsymbol{\alpha}$ in V such that $h(\boldsymbol{\alpha}) \leq \theta$. The *essential minimum* of V is defined as follow:

$$\hat{\mu}_{\mathrm{ess}}(V) := \inf\left\{\theta > 0 \mid \overline{V(\theta)}^{\mathrm{Zar}} = V\right\} ,$$

where $\overline{V(\theta)}^{\mathrm{Zar}}$ is the Zariski closure of $V(\theta)$. In [12, theorem 5.2] and [13, theorem 1.10][a], S. Zhang proves the following inequalities:

$$\frac{\hat{h}(V)}{(\dim(V)+1)\deg(V)} \leq \hat{\mu}_{\mathrm{ess}}(V) \leq \frac{\hat{h}(V)}{\deg(V)}.$$

[a]In [5, Corollaire 3.2], there is also a simpler proof in the case of abelian varieties, directly adaptable to $\mathbb{G}_m^n$.

Now consider an algebraic subtorus H of $\mathbb{G}_m^n$ and a sequence of points $(\boldsymbol{\alpha}_i)$ in $\mathbb{G}_m^n$ such that $h(\boldsymbol{\alpha}_i)$ tends to 0 when i goes to infinity (for instance $\boldsymbol{\alpha}_i = (3^{1/i}, \ldots, 3^{1/i})$). We have:

$$\hat{h}(\boldsymbol{\alpha}_i \cdot H) \leq n \deg(\boldsymbol{\alpha}_i \cdot H) \hat{\mu}_{\mathrm{ess}}(\boldsymbol{\alpha}_i \cdot H) \leq n \deg(H) h(\boldsymbol{\alpha}_i).$$

Consequently, without any assumption on the variety we cannot find a lower bound for the height depending only on the geometric degree.

Subsequently we will need a lower bound for the number of prime numbers lying in $[N/2, N]$, for some parameter N. Let's recall the lemma III.2 of [10].

Lemma 3.1. *For any real x we denote $\pi(x)$ the number of prime numbers less than or equal to x. For any integer $N \geq 2$ we have:*

$$\pi(N) - \pi(N/2) \geq 0.23 \frac{N}{\log N}.$$

Proof. The corollary 3 of [11] gives us:

$$\forall x \geq 20.5, \quad \pi(2x) - \pi(x) > \frac{3x}{5 \log x}$$

We deduce:

$$\pi(N) - \pi(N/2) > 0.3 \frac{N}{\log(N/2)} > 0.3 \frac{N}{\log N}.$$

Then, for any $N \geq 41$, we have the desired result and a numerical checking for small values of N allows us to conclude. □

4. Transcendence

Proposition 4.1. *Let V be a geometrically irreducible hypersurface of $\mathbb{G}_m^n$. Let L and T be two integers such that*

$$L + 1 \geq n \deg(V) T^2 \quad \textit{and} \quad T \geq 10n.$$

Then, for any real $\theta \leq 10^{-1} \frac{T}{L}$, there exists a nonzero polynomial $F \in \overline{\mathbb{Q}}[\mathbf{x}]$ of degree at most L, vanishing on $V(\theta)$ with multiplicity at least T satisfying

$$h(F) \leq \frac{1}{2}(n + 3) \log(L + 1).$$

Proof. First remark that since $L + 1 \geq n \deg(V) T^2$, the $\overline{\mathbb{Q}}$-vector space of the polynomials of $\overline{\mathbb{Q}}[\mathbf{x}]$ of degree at most L, vanishing on V with multiplicity at least T is not reduced to $\{0\}$. The following argument closely follows the proof of theorem 2.2 of [3]. Note that in *op. cit.*, the authors use the "*obstruction index of V of weight T*", denoted by $\omega(T; V)$, which is simply $T \deg(V)$ in our case, since V is a hypersurface. Let S be the $\overline{\mathbb{Q}}$-vector space of the polynomials of $\overline{\mathbb{Q}}[\mathbf{x}]$ of degree at most L, vanishing on $V(\theta)$ with multiplicity at least T. As in [3] we obtain[b]:

$$h(F) \leq \frac{\binom{L+n}{n} - \dim_{\overline{\mathbb{Q}}} S}{\dim_{\overline{\mathbb{Q}}} S}\Big((T+n)\log(L+1) + L\theta\Big) + \frac{1}{2}\log\binom{L+n}{n}.$$

By proposition 2.6 of *op. cit.* we have:

$$\frac{\binom{L+n}{n} - \dim_{\overline{\mathbb{Q}}} S}{\dim_{\overline{\mathbb{Q}}} S} \leq \frac{1}{T-1}.$$

Hence, from $L\theta \leq 10^{-1}T$ we get:

$$h(F) \leq \frac{1}{T-1}\Big((T+n)\log(L+1) + 10^{-1}T\Big) + \frac{n}{2}\log(L+1).$$

By hypothesis, we have $\log(L+1) \geq 1$, and thereby

$$h(F) \leq \frac{1.1T+n}{T-1}\log(L+1) + \frac{n}{2}\log(L+1).$$

From $T \geq 10n$, we finally get

$$h(F) \leq \frac{1}{2}(n+3)\log(L+1).$$ □

For the extrapolation, we will use an idea already introduced in [3]. The key point is the p-adic "proximity" of a p-th root of unity and 1; more precisely if p is a prime number, then the polynomial $(X-1)^p$ is congruent to $X^p - 1$ modulo p, and so are $(X-1)^{p-1}$ and $\phi_p(X)$. By evaluating these quantities at a p-th root of unity ξ, we get:

$$\forall v \mid p,\ |\xi - 1|_v \leq p^{-1/(p-1)} \leq p^{-1/p}. \tag{4.5}$$

For any $\boldsymbol{\mu} \in \mathbb{N}^n$, denote $|\boldsymbol{\mu}| := \mu_1 + \cdots + \mu_n$ the *length* of $\boldsymbol{\mu}$ and

$$D_{\boldsymbol{\mu}} := \frac{1}{\boldsymbol{\mu}!}\left(\frac{\partial}{\partial x_1}\right)^{\mu_1} \circ \cdots \circ \left(\frac{\partial}{\partial x_n}\right)^{\mu_n}$$

Where $\boldsymbol{\mu}! := \mu_1! \cdots \mu_n!$. Consequently, a polynomial F will vanish at a point $\boldsymbol{\alpha}$ with multiplicity at least T if for any $\boldsymbol{\mu} \in \mathbb{N}^n$ such that $|\boldsymbol{\mu}| \leq T-1$, we have $D_{\boldsymbol{\mu}}(F)(\boldsymbol{\alpha}) = 0$.

Proposition 4.2. *Let V be a subvariety of $\mathbb{G}_m^n$. Let T, L be integers and F be a nonzero polynomial in $\overline{\mathbb{Q}}[\mathbf{x}]$ of degree at most L, vanishing on V with multiplicity at least T.*

[b] In *op. cit.*, authors use the L_2 norm for the archimedean places, nevertheless, since the sup norm is smaller, the inequality still occurs.

For any prime number p and any $\boldsymbol{\xi} \in \ker[p]$, the vanishing order T^* of F on $\boldsymbol{\xi} \cdot V$ verifies:

$$T^*(1+\log(L+1)) \geq T\frac{\log p}{p} - h(F) - n\log(L+1) - L\hat{\mu}_{ess}(V).$$

Proof. Let us consider $\theta > \hat{\mu}_{\text{ess}}(V) = \hat{\mu}_{\text{ess}}(\boldsymbol{\xi} \cdot V)$ and let $\boldsymbol{\lambda}$ be any element of $\mathbb{N}^n$ of length $|\boldsymbol{\lambda}| = T^*$ such that $D_{\boldsymbol{\lambda}}(F)$ does not vanish on the whole of $\boldsymbol{\xi} \cdot V$. By density, there exists $\boldsymbol{\alpha} \in V$ such that $h(\boldsymbol{\alpha}) \leq \theta$ and $D_{\boldsymbol{\lambda}}(F)(\boldsymbol{\xi} \cdot \boldsymbol{\alpha}) \neq 0$. Generally speaking, if G is a polynomial with algebraic coefficients, then by the Taylor expansion at $\boldsymbol{\alpha}$ we get:

$$G(\boldsymbol{\xi} \cdot \boldsymbol{\alpha}) = \sum_{\boldsymbol{\mu} \in \mathbb{N}^n} D_{\boldsymbol{\mu}}(G)(\boldsymbol{\alpha}) \cdot (\boldsymbol{\xi} \cdot \boldsymbol{\alpha} - \boldsymbol{\alpha})^{\boldsymbol{\mu}}.$$

If v is a valuation dividing p, then $|D_{\boldsymbol{\mu}}(G)(\boldsymbol{\alpha}) \cdot (\boldsymbol{\xi} \cdot \boldsymbol{\alpha} - \boldsymbol{\alpha})^{\boldsymbol{\mu}}|_v$ is bounded from above by

$$|G|_v \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\}^{\deg(G)-|\boldsymbol{\mu}|} \max\{|\alpha_1\xi_1 - \alpha_1|_v, \ldots, |\alpha_n\xi_n - \alpha_n|_v\}^{|\boldsymbol{\mu}|}$$

so, by inequality (4.5):

$$|D_{\boldsymbol{\mu}}(G)(\boldsymbol{\alpha}) \cdot (\boldsymbol{\xi} \cdot \boldsymbol{\alpha} - \boldsymbol{\alpha})^{\boldsymbol{\mu}}|_v \leq |G|_v \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\}^{\deg(G)} p^{-|\boldsymbol{\mu}|/p}.$$

Note that $D_{\boldsymbol{\mu}} \circ D_{\boldsymbol{\lambda}} = aD_{\boldsymbol{\mu}+\boldsymbol{\lambda}}$ for some rational integer a, since v is a finite place, therefore we have $|D_{\boldsymbol{\mu}} \circ D_{\boldsymbol{\lambda}}(F)|_v \leq |F|_v$. Hence, by applying the previous inequality to $G = D_{\boldsymbol{\lambda}}(F)$ we get:

$$\begin{aligned} |D_{\boldsymbol{\lambda}}(F)(\boldsymbol{\xi} \cdot \boldsymbol{\alpha})|_v &\leq \max_{|\boldsymbol{\mu}| \geq T-T^*} |D_{\boldsymbol{\mu}} \circ D_{\boldsymbol{\lambda}}(F)(\boldsymbol{\alpha})(\boldsymbol{\xi} \cdot \boldsymbol{\alpha} - \boldsymbol{\alpha})^{\boldsymbol{\mu}}|_v \\ &\leq |F|_v \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\}^{L-|\boldsymbol{\lambda}|} \cdot p^{-(T-T^*)/p} \end{aligned}$$

because $D_{\boldsymbol{\lambda}}(F)$ vanishes at $\boldsymbol{\alpha}$ with multiplicity greater than $T - T^*$.

Now suppose that v is an archimedian valuation. If $F(\mathbf{x}) = \sum_{|\boldsymbol{\nu}| \leq L} \mathbf{a}_{\boldsymbol{\nu}} \cdot \mathbf{x}^{\boldsymbol{\nu}}$ we have:

$$D_{\boldsymbol{\lambda}}(F)(\mathbf{x}) = \sum_{|\boldsymbol{\nu}| \leq L} \binom{\boldsymbol{\nu}}{\boldsymbol{\lambda}} \mathbf{a}_{\boldsymbol{\nu}} \cdot \mathbf{x}^{\boldsymbol{\nu}-\boldsymbol{\lambda}}.$$

Moreover, since $\sum_{\nu=1}^{L} \binom{\nu}{\lambda} = \binom{L+1}{\lambda+1} \leq (L+1)^{\lambda+1}$, the following inequality occurs:

$$\sum_{|\boldsymbol{\nu}| \leq L} \binom{\boldsymbol{\nu}}{\boldsymbol{\lambda}} = \sum_{|\boldsymbol{\nu}| \leq L} \binom{\nu_1}{\lambda_1} \cdots \binom{\nu_n}{\lambda_n} \leq (L+1)^{|\boldsymbol{\lambda}|+n}.$$

We then obtain

$$|D_{\boldsymbol{\lambda}}(F)(\boldsymbol{\xi} \cdot \boldsymbol{\alpha})|_v \leq |F|_v \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\}^{L-|\boldsymbol{\lambda}|} \cdot (L+1)^{|\lambda|+n}.$$

To sum up we have:

$$|D_{\boldsymbol{\lambda}}(F)(\boldsymbol{\xi}\cdot\boldsymbol{\alpha})|_v \leq \begin{cases} |F|_v \max\{1,|\alpha_1|_v,\ldots,|\alpha_n|_v\}^{L-|\boldsymbol{\lambda}|} & \text{if } v\nmid\infty \\ |F|_v \max\{1,|\alpha_1|_v,\ldots,|\alpha_n|_v\}^{L-|\boldsymbol{\lambda}|}\cdot p^{-(T-T^*)/p} & \text{if } v\mid p \\ |F|_v \max\{1,|\alpha_1|_v,\ldots,|\alpha_n|_v\}^{L-|\boldsymbol{\lambda}|}\cdot (L+1)^{T^*+n} & \text{if } v\mid\infty. \end{cases}$$

We now deduce from the product formula:

$$0 \leq \frac{-(T-T^*)}{p}\log p + h(F) + (T^*+n)\log(L+1) + Lh(\boldsymbol{\alpha}),$$

then

$$T\frac{\log p}{p} - h(F) - n\log(L+1) - L\theta \leq T^*\left(\frac{\log p}{p} + \log(L+1)\right).$$

As $\theta > \hat{\mu}_{\mathrm{ess}}(V)$, the polynomial F vanishes on $\overline{V(\theta)}^{\mathrm{Zar}} = V$. Since $p \geq \log p$, by making θ tend to $\hat{\mu}_{\mathrm{ess}}(V)$, we get the desired result. □

Let us recall some properties which will be useful in the next lemma:

Lemma 4.3. *Let V be a geometrically irreducible subvariety of $\mathbb{G}_m^n$, and let G_V be its stabilizer. Let us consider an integer l, we have:*

(1) $\dim(G_V) \leq \dim(V)$ *with equality if and only if V is a translate of a subtorus.*

(2) $\deg(G_V) \leq \deg(V)^{\dim(V)-\dim(G_V)+1}$.

(3) $\deg([l]^{-1}V) = l^{n-\dim(V)}\deg(V)$ *and* $\deg([l]V) = \dfrac{l^{\dim(V)}}{|\ker[l]\cap G_V|}\deg(V)$.

(4) $|\ker[l]\cap G_V| = l^{\dim(G_V)}\left|\ker[l]\cap(G_V/G_V^0)\right|$.

(5) If $\boldsymbol{\xi}$ is a torsion point, then $\deg(\boldsymbol{\xi}\cdot V) = \deg(V)$.

Proof. Let us prove point (1). We have

$$G_V = \{\boldsymbol{\alpha}\in\mathbb{G}_m^n \mid \boldsymbol{\alpha}\cdot V = V\} = \bigcap_{\mathbf{y}\in V}\mathbf{y}^{-1}\cdot V\ ,$$

in particular $\dim(G_V) \leq \dim(V)$. If $\dim(G_V) = \dim(V)$, then, since V is irreducible, for all $\mathbf{y}\in V$ we have $G_V = \mathbf{y}^{-1}\cdot V$; therefore V is a translate of its stabilizer. Reciprocally, if H is an algebraic subgroup of $\mathbb{G}_m^n$ and $\mathbf{x}$ an element of $\mathbb{G}_m^n$ such that $V = \mathbf{x}\cdot H$, then

$$G_V = \bigcap_{\mathbf{y}\in V}\mathbf{y}^{-1}\cdot V = \bigcap_{\mathbf{y}\in V}(\mathbf{y}^{-1}\mathbf{x})\cdot H = \bigcap_{\mathbf{h}\in H}\mathbf{h}^{-1}\cdot H = G_H = H.$$

Therefore $\dim(G_V) = \dim(H) = \dim(V)$.

For (4) and (2) one can see [1], lemme 2.1 and above this lemma respectively. For (3) and (5), see [6] proposition 2.1. □

Lemma 4.4. *Let V be a proper geometrically irreducible subvariety of $\mathbb{G}_m^n$ and let N be an integer, $N \geq 2$. If V is not a translated subtorus of $\mathbb{G}_m^n$ and if its stabilizer G_V is connected, then*

$$\deg\left(\bigcup_{\substack{p\in[N/2,N]\\ p \text{ prime}}}[p]^{-1}[p]\cdot V\right) \geq 0.23\frac{N}{\log N}\left(\left(\frac{N}{2}\right)^{n-\dim(G_V)}-1\right)\deg(V).$$

Proof. For any prime p in $[N/2, N]$ we have:

$$[p]^{-1}[p]\cdot V = \bigcup_{\boldsymbol{\xi}_p\in\ker[p]}\boldsymbol{\xi}_p\cdot V.$$

Moreover, all translated $\boldsymbol{\xi}_p \cdot V$ of V are geometrically irreducible of degree $\deg(V)$ (from point (5) of lemma 4.3). Let us now investigate under which conditions two of these translated of V are equal. Let us consider two distinct prime numbers p, p' in $[N/2, N]$ and two torsion points $(\boldsymbol{\xi}_p, \boldsymbol{\xi}_{p'})$ in $\ker[p]\times\ker[p']$ such that $\boldsymbol{\xi}_p\cdot V = \boldsymbol{\xi}_{p'}\cdot V$. So we have $\boldsymbol{\xi}_p^{-1}\boldsymbol{\xi}_{p'} \in G_V$, and then, since p and p' are coprime, $\boldsymbol{\xi}_p$ and $\boldsymbol{\xi}_{p'}$ belong to the stabilizer G_V of V. Therefore we have:

$$\begin{aligned}\deg\left(\bigcup_{\substack{p\in[N/2,N]\\ p \text{ prime}}}[p]^{-1}[p]\cdot V\right) &\geq \sum_{\substack{p\in[N/2,N]\\ p \text{ prime}}}\deg\left(\bigcup_{\boldsymbol{\xi}_p\in\ker[p]\setminus G_V}\boldsymbol{\xi}_p\cdot V\right)\\ &\geq \sum_{\substack{p\in[N/2,N]\\ p \text{ prime}}}\left(\deg\left(\bigcup_{\boldsymbol{\xi}_p\in\ker[p]}\boldsymbol{\xi}_p\cdot V\right)-\deg(V)\right).\end{aligned}$$

Since by hypothesis G_V is connected, (then $|G_V/G_V^0| = 1$), by points (3) and (4) of lemma 4.3 we have, for any prime $p \in [N/2, N]$:

$$\begin{cases} |\ker[p]\cap G_V| = p^{\dim(G_V)}\left|\ker[p]\cap(G_V/G_V^0)\right| = p^{\dim(G_V)}\\ \deg\left([p]^{-1}[p]\cdot V\right) = p^{n-\dim(G_V)}\deg(V).\end{cases}$$

Then:

$$\begin{aligned}\deg\left(\bigcup_{\substack{p\in[N/2,N]\\ p \text{ prime}}}[p]^{-1}[p]\cdot V\right) &\geq \sum_{\substack{p\in[N/2,N]\\ p \text{ prime}}}\left(p^{n-\dim(G_V)}-1\right)\deg(V)\\ &\geq \left(\pi(N)-\pi(N/2)\right)\left(\left(\frac{N}{2}\right)^{n-\dim(G_V)}-1\right)\deg(V).\end{aligned}$$

To conclude, we only need to remark that, from lemma 3.1, we have:

$$\pi(N) - \pi(N/2) \geq 0.23 \frac{N}{\log N}. \qquad \square$$

5. Conclusion

As in [2], we will see that one can assume that the stabilizer G_V of V is connected, even if it entails a smaller constant. So we first prove the following result.

Theorem 5.1. *Let V be a geometrically irreducible hypersurface of $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$ of degree ω. If V is not a translated subtorus of $\mathbb{G}_m^n$ and if the stabilizer of V is connected, then*

$$\hat{\mu}_{ess}(V) \geq \frac{3 \cdot 10^{-12}}{n^8 \omega} \frac{(\log(n \log \omega'))^{2+2/(n-s-1)}}{(\log \omega')^{1+4/(n-s-1)}} \quad ,$$

where $\omega' := \max\{n\omega, 16\}$ and $s = \dim(G_V)$.

Since the case $n = 1$ is empty, we then assume from now $n \geq 2$. Let us consider three parameters

$$\begin{cases} N := 2 \left(c_1 \dfrac{(\log \omega')^2}{\log(n \log \omega')} \right)^{1/(n-s-1)} & (c_1 := 100 n c_0^2) \\ T := \left[c_0 \dfrac{N \log \omega'}{\log(n \log \omega')} \right] & (c_0 := 10(n+1)(n-s-1)) \\ L := n\omega T^2 \end{cases}$$

Claim 5.2. *We have the following inequalities:*

(1) $\frac{3}{2}(n+1)\log(L+1) < 0.84 T \frac{\log N}{N}$
(2) $L < \frac{0.22}{2^{n-s}} \frac{N^{n-s+1}}{\log N} \omega$.

This technical claim will be proved in the appendix. We proceed by contradiction, assume that

$$L \hat{\mu}_{\text{ess}}(V) < T \frac{\log N}{10N}. \tag{5.12}$$

Applying proposition 4.1 with $\theta := \frac{T \log N}{10NL} > \hat{\mu}_{\text{ess}}(V)$, we get a nonzero polynomial $F \in \overline{\mathbb{Q}}[\mathbf{x}]$ of degree at most L, vanishing on $V(\theta)$ (and therefore on V by density) with multiplicity at least T such that

$$h(F) \leq \frac{1}{2}(n+3)\log(L+1). \tag{5.13}$$

Denote T^* the order of vanishing of F on $\boldsymbol{\xi} \cdot V$, where $\boldsymbol{\xi}$ runs over $\ker[p]$ and the prime number p runs over $[N/2, N]$. From proposition 4.2 we get[c]:

$$T^*(1+\log(L+1)) \geq 0.94T\frac{\log N}{N} - h(F) - n\log(L+1) - L\hat{\mu}_{\mathrm{ess}}(V)$$

so, by (5.12) and (5.13)

$$T^*(1+\log(L+1)) \geq 0.84T\frac{\log N}{N} - \frac{3}{2}(n+1)\log(L+1).$$

Hence, from claim 5.2, we get $T^* > 0$, in other words, for any prime p in $[N/2, N]$ and any $\boldsymbol{\xi}$ in $\ker[p]$, the polynomial F vanishes on $\boldsymbol{\xi} \cdot V$. Since G_V is connected and $N \geq 2$, lemma 4.4 give us:

$$L \geq \deg(F) \geq 0.23\frac{N}{\log N}\left(\left(\frac{N}{2}\right)^{n-s} - 1\right)\omega.$$

Thus, from $1 \leq \left(\frac{N}{2}\right)^{n-s}\frac{1}{23}$,

$$L \geq \frac{0.22}{2^{n-s}}\frac{N^{n-s+1}}{\log N}\omega.$$

Using claim 5.2, we get a contradiction. So the hypothesis (5.12) is false and:

$$\begin{aligned}\hat{\mu}_{\mathrm{ess}}(V) &\geq T\frac{\log N}{10NL} = \frac{\log N}{10Nn\omega T}\\ &\geq \frac{\log N}{10N^2n\omega}\frac{\log\left(n\log\omega'\right)}{c_0\log\omega'}\ ,\end{aligned}$$

since $\log N \geq \frac{1}{n-s-1}\log(n\log\omega')$, it follows therefrom that

$$\hat{\mu}_{\mathrm{ess}}(V) \geq \frac{1}{40nc_0c_1^{2/(n-s-1)}(n-s-1)}\frac{1}{\omega}\frac{\left(\log\left(n\log\omega'\right)\right)^{2+2/(n-s-1)}}{(\log\omega')^{1+4/(n-s-1)}}.$$

We still have to compute the constant. Let us denote $x := n-s-1$. Since $c_0 = 10(n+1)x$ and $c_1 = 100nc_0^2$, we have:

$$\begin{aligned}40nc_0c_1^{2/x}x &= 40nc_0(100nc_0^2)^{2/x}x\\ &= 4\cdot 10^{1+4/x}n^{1+2/x}c_0^{1+4/x}x\\ &= 4\cdot 10^{1+4/x}n^{1+2/x}\big(10(n+1)x\big)^{1+4/x}x\\ &= 4\cdot 10^{2+8/x}n^{1+2/x}(n+1)^{1+4/x}x^{2+4/x}.\end{aligned}$$

If we denote by $f(x)$ this last expression, and if $x \in \{2, \ldots, n\}$ we have:

$$f(x) \leq 4\cdot 10^6n^2(n+1)^3n^2e^4 \leq 4\cdot 10^{10}n^3(n+1)^5 = f(1).$$

[c] we have to write 0.94 instead of 1 because if $(p, N) = (2, 3)$, we do not have $\log(p)/p \geq \log(N)/N$.

So the most unfavourable value of $n-s-1$ is 1, in which case we have (since $n+1\geq 3$):

$$f(1)=4\cdot 10^{10}n^3(n+1)^5\leq 3.1\cdot 10^{11}n^8.$$

Then

$$\hat{\mu}_{\mathrm{ess}}(V)\geq \frac{3\cdot 10^{-12}}{n^8}\frac{1}{\omega}\frac{(\log(n\log\omega'))^{2+2/(n-s-1)}}{(\log\omega')^{1+4/(n-s-1)}}.$$

□

We are now able to deduce theorem 2.4 from theorem 5.1.

Proof of theorem 2.4. Proposition 2.4 of [2] gives[d] us an irreducible hypersurface V_1 of degree $\leq n^2\omega$ whose stabilizer is connected and such that

$$\hat{h}(V_1)\leq\hat{h}(V)\quad\text{and}\quad\dim G_{V_1}=\dim G_V.$$

Consider the irreducible hypersurface V_1 given by proposition 2.4 of [2]. Its stabilizer is connected and $\deg(V_1)\leq n^2\omega$, moreover

$$\hat{h}(V_1)\leq\hat{h}(V)\quad\text{and}\quad\dim G_{V_1}=\dim G_V.$$

Denote ω_1 the degree of V_1, by applying theorem 5.1 we get:

$$\hat{\mu}_{\mathrm{ess}}(V_1)\geq \frac{3\cdot 10^{-12}}{n^8\omega_1}\frac{(\log(n\log\omega_1'))^{2+2/(n-s-1)}}{(\log\omega_1')^{1+4/(n-s-1)}}$$

where $\omega_1':=\max\{n\omega_1,16\}$. Remark that, since

$$\omega_1'\leq\max\{n^3\omega,16\}\leq n^2\max\{n\omega,16\}=n^2\omega'\ ,$$

we have:

$$(\log\omega_1')^{1+4/(n-s-1)}\leq(2\log n+\log\omega')^{1+4/(n-s-1)}\leq 3^5\log\omega'^{1+4/(n-s-1)}.$$

We then conclude by applying the following inequality:

$$\hat{h}(V)\geq\hat{h}(V_1)\geq\omega_1\hat{\mu}_{\mathrm{ess}}(V_1).$$

□

[d] Note that this proposition is stated for a $\mathbb{Q}$-irreducible subvariety, however the proof is the same for $\overline{\mathbb{Q}}$-irreducible ones.

6. Appendix: proof of the claim 5.2

We will frequently use the following inequality.

$$\frac{x^a}{(\log x)^b} \geq \left(\frac{ea}{b}\right)^b. \tag{6.18}$$

which holds for all $a, b \in \mathbb{R}^+$ and all $x > 1$.

From $N \geq 2$ and $c_0 \geq 10n$ we get

$$T \geq \left[20\frac{n\log\omega'}{\log(n\log\omega')}\right]$$

then, using (6.18) with $a = b = 1$ and $x = n\log\omega'$, we obtain $T \geq [20e] \geq 27$. Moreover,

$$\begin{aligned} N^{n-s-1} &= 2^{n-s-1}\cdot 100n\frac{(c_0\log\omega')^2}{\log(n\log\omega')} \\ &\geq 100\frac{(n\log\omega')^{0.01}}{\log(n\log\omega')}\,(c_0\log\omega')^{1.99} \end{aligned}$$

and so, using (6.18) with $a = 0.01$, $b = 1$ and $x = n\log\omega'$, we get $N^{n-s-1} \geq (c_0\log\omega')^{1.99}$. Finally

$$\begin{aligned} \log N &< \log(2c_1/n^2) + 2\log(n\log\omega') \\ &< \log(2\cdot 10^4 n) + 2\log(n\log\omega'). \end{aligned}$$

From $n \geq 2$ we deduce $\log N < 11\log(n\log\omega')$.

We can now prove the first point of claim 5.2. Since $\log N \geq \frac{1.99}{n-s-1}\log(c_0\log\omega')$, we have:

$$\begin{aligned} \frac{\log L}{\log N} &\leq \frac{\log\omega' + 2\log N + 2\log(c_0\log\omega')}{\log N} \\ &\leq \frac{(n-s-1)}{1.99}\frac{\log\omega'}{\log(n\log\omega')} + 2 + \frac{2(n-s-1)}{1.99} \\ &\leq (n-s-1)\left(\frac{\log\omega'}{\log(n\log\omega')} + 3.02\right) \\ &\leq (n-s-1)\frac{\log\omega'}{\log(n\log\omega')}\left(1 + 3.02(1+e^{-1})\right). \end{aligned}$$

Hence

$$N\times\frac{3}{2}(n+1)\log L \leq N\times\frac{3}{2}(n+1)(n-s-1)\frac{\log\omega'}{\log(n\log\omega')}\cdot 5.2\log N.$$

To conclude, we just have to remark that since $T \geq 27$, we have $\frac{T+1}{T} \leq \frac{28}{27}$ and $\log(L+1) \leq 1.001\log L$, so:

$$N\times\frac{3}{2}(n+1)\log(L+1) < 0.84T\log N.$$

We now prove the second point of claim 5.2. We have:

$$\frac{0.22}{2^{n-s}}\frac{N^{n-s+1}}{\log N}\omega = \frac{11\cdot 10^{-2}}{\log N}\left(\frac{N}{2}\right)^{n-s-1} N^2\omega.$$

Hence, since $\log N < 11\log(n\log\omega')$:

$$\begin{aligned}\frac{0.22}{2^{n-s}}\frac{N^{n-s+1}}{\log N}\omega &= \frac{11\cdot 10^{-2}}{\log N}\left(\frac{N}{2}\right)^{n-s-1} N^2\omega \\ &> \frac{10^{-2}}{\log(n\log\omega')}\left(c_1\frac{(\log\omega')^2}{\log(n\log\omega')}\right)N^2\omega \\ &= n\omega\left(\frac{c_0 N\log\omega'}{\log(n\log\omega')}\right)^2 \geq L.\end{aligned}$$

□

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