ADDENDA TO “FOUNDATIONS OF GARSIDE THEORY”

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with

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Chapter I: Examples

Skipped proofs

(none)

Solution to selected exercises

(none)

Chapter II: Preliminaries

Skipped proofs

Proposition II.1.18 (collapsing invertible elements).— For $\mathcal{C}$ a left-cancellative category, the following conditions are equivalent:

(i) The equivalence relation $\equiv =$ is compatible with composition, in the sense that, if $g_1g_2$ is defined and $g'_i \equiv = g_i$ holds for $i = 1, 2$, then $g'_1g'_2$ is defined and $g'_1g'_2 \equiv = g_1g_2$ holds;

(ii) The family $\mathcal{C}^\equiv(x, y)$ is empty for $x \neq y$ and, for all $g, g'$ sharing the same source, $g \equiv = g'$ implies $g \equiv = g'$;

(iii) The family $\mathcal{C}^\equiv(x, y)$ is empty for $x \neq y$ and we have

$\forall x, y \in \text{Obj}(\mathcal{C}) \ \forall g \in \mathcal{C}(x, y) \ \forall \epsilon \in \mathcal{C}^\equiv(x, x) \ \exists \epsilon' \in \mathcal{C}^\equiv(y, y) \ (\epsilon g = g\epsilon')$.

When the above conditions are satisfied, the equivalence relation $\equiv =$ is compatible with composition, and the quotient-category $\mathcal{C}/\equiv =$ has no nontrivial invertible element.
Proof. Assume (i). Let $\epsilon \in C^\ast(x,y)$. Then $\epsilon = \ast 1_x$ and $1_y = \ast 1_y$ are satisfied, and $\epsilon 1_y$ is defined. By (i), $1_x 1_y$ must be defined as well, which is possible only for $x = y$.

Let now $g \in C(x,y)$ and $\epsilon \in C^\ast(x,x)$. Then $\epsilon = \ast 1_x$ and $g = \ast g$ are satisfied, and $\epsilon g$ is defined. By (i), we must have $g = \ast eg$, that is, there must exist $\epsilon'$ in $C(y,y)$ satisfying $eg = g\epsilon'$. So (i) implies (iii).

Assume now (ii). Let $g \in C(x,y)$ and $\epsilon \in C^\ast(x,x)$. Then $g = \ast eg$ holds, as we can write $\epsilon g = eg 1_y$. By (ii), we deduce $g = \ast eg$, so, as above, there must exist $\epsilon'$ in $C(y,y)$ satisfying $eg = g\epsilon'$. So (ii) implies (iii).

Assume now (iii). Let $g_1 \in C(x,y)$, $g_2 \in C(y,z)$, and assume $g_1' = \ast g_1$ and $g_2' = \ast g_2$. By assumption, there exists $\epsilon_i$ in $C^\ast$ satisfying $g_i' = g_i \epsilon_i$ for $i = 1, 2$.

Applying (II.1.19) to $g_2$ and $\epsilon_1$, we deduce that there exists $\epsilon_1' \in C^\ast(z,z)$ satisfying $\epsilon_1 1 g_2 = g_2 \epsilon_1'$. We deduce $g_1' g_2' = g_1 g_2 \epsilon_1' \epsilon_2$, whence $g_1' g_2' = \ast g_1 g_2$. So (iii) implies (i).

Next, assume (iii) again, and $g' = \ast g$. By definition, there exist $\epsilon, \epsilon'$ satisfying $g' = g\epsilon \epsilon'^{-1}$, and we deduce $g' = g \epsilon \epsilon'^{-1}$, whence $g' = \ast g$. So (iii) implies (ii).

Finally, assume that (i), (ii), and (iii) are satisfied, we have $g_1 = \ast g_1'$ and $g_2 = \ast g_2'$, and $g_1 g_2$ and $g_1' g_2'$ exist. By assumption, there exist for $i = 1, 2$ invertible elements $\epsilon_i, \epsilon'_i$ satisfying $\epsilon_i g_i = g'_i \epsilon_i$. Let $x$ and $y$ be the source and target of $g_2$.

By construction, $\epsilon_1^{-1} \epsilon_2$ belongs to $C^\ast(x,x)$. Applying (iii), we deduce the existence of $\epsilon$ in $C^\ast(y,y)$ satisfying $\epsilon_1^{-1} \epsilon_2 g_2 = g_2 \epsilon$. Then we obtain

$$\epsilon_1^{-1} g_1 g_2 = \epsilon_1^{-1} g_1 g_2 \epsilon = \epsilon_1^{-1} g_1 \epsilon_1^{-1} \epsilon_2 g_2 = g_1' g_2' \epsilon_2,$$

which shows that $g_1 g_2 = \ast g_1' g_2'$ is true. So $\ast = \ast$ is a congruence, and there exists a well defined quotient category $C/\ast = \ast$, obtained from $C$ by identifying elements that are $\ast = \ast$-equivalent. In the current case, according to (ii), distinct objects of $C$ are never connected by invertible elements, so the collapsing only involves the elements.

Then, by construction, the category $C/\ast = \ast$ has no nontrivial invertible element. □

**Solution to selected exercises**

**Exercice 4 (atom).—** Assume that $C$ is a left-cancellative category. Show that, for $n \geq 1$, every element $g$ of $C$ satisfying $ht(g) = n$ admits a decomposition into a product of $n$ atoms.

**Solution.** By Proposition II.2.48, there exists a decomposition $(g_1, \ldots, g_n)$ of $g$ consisting of $n$ non-invertible entries. The assumption that $g_i$ is non-invertible implies $ht(g_i) \geq 1$ for every $i$. Hence we must have $ht(g_i) = 1$ for each $i$, so each factor $g_i$ is an atom. Hence $g$ admits a decomposition as a product of $n$ atoms.

**Exercice 5 (unique right-mcm).—** Assume that $C$ is a left-cancellative category that admits right-mcms. Assume that $f, g$ are elements of $C$ that admit a common right-multiple and any two right-mcms of $f$ and $g$ are $\ast$-equivalent. Show that every right-mcm of $f$ and $g$ is a right-lcm of $f$ and $g$.

**Solution.** Let $h$ be a right-mcm of $f$ and $g$, and $\hat{h}$ be a common right-multiple of $f$ and $g$. Then $\hat{h}$ is a right-multiple of some right-mcm $h'$ of $f$ and $g$. By assumption, $h' = \ast h$ holds, hence $\hat{h}$ is a right-multiple of $h$ as well. Hence $h$ is a right-lcm of $f$ and $g$. 

**Exercise 6 (right-gcd to right-mcm).**—Assume that $\mathcal{C}$ is a cancellative category that admits right-gcds. $f, g$ are elements of $\mathcal{C}$ and $h$ is a common right-multiple of $f$ and $g$. Show that there exists a right-mcm $h_0$ of $f$ and $g$ such that every common right-multiple of $f$ and $g$ that left-divides $h$ is a right-multiple of $h_0$.

**Solution.** (See Figure 1.) Write $h = fg = g\hat{f}$, and let $\hat{h}$ be a right-gcd of $\hat{f}$ and $\hat{g}$. By definition $\hat{h}$ right-divides $\hat{f}$ and $\hat{g}$, so there exist $\hat{f}', \hat{g}'$ satisfying $\hat{f} = \hat{f}'\hat{h}$ and $\hat{g} = \hat{g}'\hat{h}$. Then we have $fg\hat{h} = fg = g\hat{f} = g\hat{f}'\hat{h}$, whence $fg' = gf'$ by right-cancelling $\hat{h}$. Assume $fg'' = gf'' \preceq h$, say $hg''h''$. By left-cancelling $f$, we deduce $\hat{g} = g''\hat{h}$ and, similarly, $\hat{f} = f''\hat{h}$. So $h''$ is a common right-divisor of $\hat{f}$ and $\hat{g}$, hence it is a right-divisor of $h$, that is, there exists $h'$ satisfying $h = h'h''$. This implies $fg''h'' = f\hat{g}'$, whence $g''h'' = \hat{g}$ by left-cancelling $f$, and, finally, $h'' = fg'' \preceq fg = h$. So every common right-multiple of $f$ and $g$ that left-divides $h$ is a right-multiple of $fg'$.

Now assume $fg'' = gf'' \preceq fg'$. A fortiori, we have $fg'' = gf'' \preceq h$, so the above result implies $fg' \preceq fg''$, whence $fg'' = fg'$. So $fg'$ is a right-mcm of $f$ and $g$.

![Figure 1. Solution to Exercise 6](image)

**Exercise 8 (conditional right-lcm).**—Assume that $\mathcal{C}$ is a left-cancellative category. (i) Show that every left-lcm of $fg_1$ and $fg_2$ (if any) is of the form $fg$ where $g$ is a left-gcd of $g_1$ and $g_2$. (ii) Assume moreover that $\mathcal{C}$ admits conditional right-lcms. Show that, if $g$ is a left-gcd of $g_1$ and $g_2$ and $fg$ is defined, then $fg$ is a left-gcd of $fg_1$ and $fg_2$.

**Solution.** (i) Since $f$ left-divides $fg_1$ and $fg_2$, it left-divides every left-gcd of $fg_1$ and $fg_2$, so the latter can be written $fg$. Now assume that $fg$ is a left-gcd of $fg_1$ and $fg_2$. As $\mathcal{C}$ is left-cancellative, $fg \preceq fg_1$ implies $g \preceq g_1$. Next, assume that $h$ left-divides $g_1$ and $g_2$. Then $fh$ left-divides $fg_1$ and $fg_2$, implying $fh \preceq fg$, whence $h \preceq g$. So $g$ is a left-gcd of $g_1$ and $g_2$.

(ii) It is clear that $fg$ is a common left-divisor of $fg_1$ and $fg_2$. Conversely, assume that $h$ is a common left-divisor of $fg_1$ and $fg_2$. By assumption, $f$ and $h$ admit $fg_1$ as a common right-multiple, so they admit a right-lcm, say $fh'$. By assumption, we have $fh' \preceq fg_1$ and $fh' \preceq fg_2$, whence $h' \preceq g_1$ and $h' \preceq g_2$ by left-cancelling $f$. This in turn implies $h' \preceq g$ since $g$ is a left-gcd of $g_1$ and $g_2$. Hence we deduce $h \preceq fh' \preceq fg$, which shows that $fg$ is a left-gcd of $fg_1$ and $fg_2$.

**Exercise 9 (left-coprime).**—Assume that $\mathcal{C}$ is a left-cancellative category. Say that two elements $f, g$ of $\mathcal{C}$ sharing the same source $x$ are left-coprime if $lx$ is a left-gcd of $f$ and $g$. Assume that $g_1, g_2$ are elements of $\mathcal{C}$ sharing the same source, and $fg_1$ and $fg_2$ are defined. Consider the properties (i) The elements $g_1$ and $g_2$...
are left-coprime; (ii) The element $f$ is a left-gcd for $fg_1$ and $fg_2$. Show that (ii) implies (i) and that, if $C$ admits conditional right-lcms, (ii) implies (i). [Hint: Use Exercise 8.]

Solution. If $g$ is a non-invertible common left-divisor of $g_1$ and $g_2$, then $fg$ is a non-invertible common left-divisor of $fg_1$ and $fg_2$, so clearly (ii) implies (i).

Conversely, assume that $C$ admits conditional right-lcms and $g_1, g_2$ are left-coprime. By definition, $1_x$ is a left-gcd of $g_1$ and $g_2$. Hence, by Exercise 8, $f$ is a left-gcd of $fg_1$ and $fg_2$. So (i) implies (ii) in this case.

Exercise 10 (subgroupoid).— Let $M = \langle a, b, c, d \mid ab = bc = cd = da \rangle^*$ and $\Delta = ab$. (i) Check that $M$ is a Garside monoid with Garside element $\Delta$. (ii) Let $M_1 (\text{resp. } M_2)$ be the submonoid of $M$ generated by $a$ and $c$ (resp. $b$ and $c$). Show that $M_1$ and $M_2$ are free monoids of rank 2 with intersection reduced to \{1\}. (iii) Let $G$ be the group of fractions of $M$. Show that the intersection of the subgroups of $G$ generated by $M_1$ and $M_2$ is not \{1\}.

Solution. (ii) No word in $\{a, c\}^*$ is eligible for any of the defining relations of $M$, so two distinct such words represent distinct elements of $M_1$. (iii) In $G$, we have $c^{-1}a = db^{-1}$ but $db^{-1}$ cannot be expressed as $f^{-1}g$ with $f, g$ in $M_2$ since, otherwise, $M_2$ would not be free.

Exercise 11 (weakly right-cancellative).— Say that a category $C$ is weakly right-cancellative if $gh = h$ implies that $g$ is invertible. (i) Observe that a right-cancellative category is weakly right-cancellative; (ii) Assume that $C$ is a left-cancellative category. Show that $C$ is weakly right-cancellative if and only if, for all $f, g$ in $C$, the relation $f \preceq g$ is equivalent to the conjunction of $f \succeq g$ and "$g = gf$ holds for no $g'$ in $C^n$.

Solution. (ii) The conjunction of $f \preceq g$ and "$g = gf$ holds for no $g'$ in $C^n$ always implies $f \succeq g$. Assume that $C$ is weakly right-cancellative and $g \succeq f$ holds. Then we have $g = gf$ for some $g' \notin C^n$. Assume $g = gf$ if $g''$ is invertible, we deduce $f = g''^{-1}g = g''^{-1}gf$. The assumption that $C$ is weakly right-cancellative implies that $g''^{-1}g$ is invertible, hence that $g''$ is invertible. So $f \preceq g$ holds. Conversely, assume that $f \preceq g$ is equivalent to the conjunction of $f \succeq g$ and "$g = gf$ holds for no $g'$ in $C^n$. Assume $gh = h$. If $x$ is the source of $h$, we have $1_xh = h$ and $1_x$ is invertible. Then the assumption implies that $h \preceq h$ fails, which, as $h \equiv h$ is true, means that $h = gh$ holds for no non-invertible $g$.

Exercise 13 (increasing sequences).— Assume that $C$ is a left-cancellative. For $S$ included in $C$, put $\text{Div}_S(h) = \{ f \in C \mid \exists g \in S : fg = h \}$. Show that the restriction of $\succeq$ to $S$ is well-founded (that is, admits no infinite descending sequence) if and only if, for every $g$ in $S$, every strictly increasing sequence in $\text{Div}_S(g)$ with respect to left-divisibility is finite.

Solution. Assume that $S$ is not right-Noetherian. Let $g_0, g_1, \ldots$ be an infinite descending sequence with respect to proper right-divisibility in $S$. For each $i$, choose a (necessarily non-invertible) element $f_i$ satisfying $g_{i-1} = f_i g_i$. Then we have $g_0 = f_1 g_1 = (f_1 f_2) g_2 = \ldots$, and the sequence $1_x (x$ the source of $g_0), f_1, f_1 f_2, \ldots$ is $\prec$-increasing in $\text{Div}_S(g_0)$.

Conversely, assume that $g_0$ lies in $S$ and $h_1 \prec h_2 \prec \ldots$ is a strictly increasing sequence in $\text{Div}_S(g_0)$. Then, for each $i$, there exists a non-invertible element $f_i$
satisfying $h_if_i = h_{i+1}$. On the other hand, as $h_i$ belongs to $\text{Div}(g_i)$, there exists $g_i$ in $S$ satisfying $h_ig_i = g_0$. We find $g_0 = h_ig_i = h_{i+1}g_{i+1} = h_if_{i+1}$. By left-cancelling $h_i$, we deduce $g_i = f_{i+1}$, hence $g_{i+1}$ is a proper right-divisor of $g_i$ for each $i$. So the sequence $g_0, g_1, \ldots$ witnesses that $S$ is not right-Noetherian.

**Exercice 15 (left-generating).**— Assume that $C$ is a left-cancellative category that is right-Noetherian. Say that a subfamily $S$ of $C$ left-generates (resp. right-generates) $C$ if every non-invertible element of $C$ admits at least one non-invertible left-divisor (resp. right-divisor) belonging to $S$. (i) Show that $C$ is right-generated by its atoms. (ii) Show that, if $S$ is a subfamily of $C$ that left-generates $C$, then $S \cup C^\times$ generates $C$. (iii) Conversely, show that, if $S \cup C^\times$ generates $C$ and $C^\times S \subseteq S^\bullet$ holds, then $S$ left-generates $C$.

**Solution.** (ii) Assume that $S$ left-generates $C$. Let $g$ be an arbitrary element of $C$. If $g$ is invertible, $g$ belongs to $C^\times$. Otherwise, by assumption, there exist a non-invertible element $g_1$ in $S$ and $g'$ in $C$ satisfying $g = g_1g'$. If $g'$ is invertible, $g$ belongs to $SC^\times$. Otherwise, there exist a non-invertible element $g_2$ in $S$ and $g''$ satisfying $g' = g_2g''$, and so on. By Proposition II.2.28, the sequence $g_1, g_1g_2, \ldots$, which is increasing with respect to proper left-divisibility and lies in $\text{Div}(g)$, must be finite, yielding $\ell$ and $g = g_1 \cdots g_\ell$ with $g_1, \ldots, g_\ell$ in $S$ and $\epsilon$ in $C^\times$.

(iii) Assume that $S \cup C^\times$ generates $C$ and $C^\times S \subseteq S^\bullet$ holds. Let $g$ be a non-invertible element of $C$. Let $(g_1, \ldots, g_p)$ be a decomposition of $g$ such that $g_i$ lies in $S \cup C^\times$ for each $i$. As $g$ is not invertible, there exists $i$ such that $g_i$ is not invertible. Assume that $i$ is minimal with this property. Then $g_1 \cdots g_{i-1}$ is invertible and, as $C^\times S \subseteq S^\bullet$ holds, there exists $g'$ in $S \setminus C^\times$ and $\epsilon$ in $C^\times$ satisfying $g_1 \cdots g_i = g'\epsilon$. Then $g'$ is a non-invertible element of $S$ left-dividing $g$. Hence $S$ left-generates $C$.

**Exercice 20 (equivalence).**— Assume that $(S, R)$ is a category presentation. Say that an element $s$ of $S$ is $R$-right-invertible if $sw \equiv_R^\bullet \varepsilon_x$ ($x$ the source of $s$) holds for some $w$ in $S^\bullet$. Show that a category presentation $(S, R)$ is complete with respect to right-reversing if and only if, for all $u, v$ in $S^\bullet$, the following are equivalent: (i) $u$ and $v$ are $R$-equivalent (that is, $u \equiv_R v$ holds), (ii) $wv \equiv_R v'w$ holds for some $R$-equivalent paths $u', v'$ in $S^\bullet$ all of which entries are $R$-right-invertible.

**Solution.** Assume that right-reversing is complete for $(S, R)$ and $u \equiv_R v$ holds. Denoting by $y$ the common target of $u$ and $v$, we have $u\varepsilon_y \equiv_R v\varepsilon_y$. Hence, by definition of completeness, there exist $u', v', w$ satisfying $wv \equiv_R v'w$, $\varepsilon_y \equiv_R u'w$, and $\varepsilon_y \equiv_R u'w$. Hence, in $C$, we have $[u']^+[w] = [v']^+[\omega] = 1_y = [v']^+[w]^+$, whence $[u']^+ = ([v']^+)^{-1}$. So $u'$ and $v'$ are $R$-equivalent and, as their classes are invertible, they must consist of invertible entries.

Conversely, assume that the condition of (ii) is satisfied, and that $u'v'$ and $v'u'$ are $R$-equivalent. By (ii), there exist $R$-equivalent $S$-paths $u_0, v_0$ such that $(\overline{uv'})\overline{(vu')} = w$ for all entries in $u_0$ and $v_0$ are invertible. By Lemma II.4.23, the reversing of $(uv')\overline{(vu')}$ to $v_0u_0$ splits into four reversings. By construction, all entries in $u_1, u_2, v_1, v_2$ are invertible. Put $w = u''v_2u_2v_1$. By assumption, we have $v_1u_2 \equiv_R u_1v_2$, whence $\overline{uv} \equiv_R \overline{vu}$. We deduce

$$u' \equiv^+ u''u''u_2v_2v_1 = u''w, \text{ and } v' \equiv^+ v''u''v_2u_2v_1 \equiv^+ u''w,$$

which means that $(u, v, u', v')$ factorizing through right-reversing. Hence right-reversing is complete for $(S, R)$. Addenda to “Foundations of Garside Theory”, by Dehornoy et al.
Exercice 22 (complete vs. cube, complemented case).— Assume that \( (\mathcal{S}, \mathcal{R}) \) is a right-complemented presentation, associated with the syntactic right-complement \( \theta \). Show the equivalence of the following three properties: (i) Right-reversing is complete for \( (\mathcal{S}, \mathcal{R}) \); (ii) The map \( \theta^\ast \) is compatible with \( \equiv^\vee \)-equivalence in the sense that, if \( u' \equiv^\vee u \) and \( v' \equiv^\vee v \) hold, then \( \theta^\ast(u', v') \) is defined if and only if \( \theta(u, v) \) is and, if so, they are \( \equiv^\vee \)-equivalent; (iii) The \( \theta \)-cube condition is true for every triple of \( \mathcal{S} \)-paths. [Hint: For (i) \( \Rightarrow \) (ii), note that, if \( \theta^\ast(u, v) \) is defined and \( u' \equiv^\vee u \) and \( v' \equiv^\vee v \) hold, then \( (u', v', \theta^\ast(u, v), \theta^\ast(v, u)) \) must be \( \equiv \)-factorable; for (ii) \( \Rightarrow \) (iii), compute \( \theta^\ast(w, u\theta^\ast(u, v)) \) and \( \theta^\ast(w, v\theta^\ast(v, u)) \); for (iii) \( \Rightarrow \) (i), use Lemma II.4.61.]

Solution. (i) \( \Rightarrow \) (ii) Assume that \( \theta^\ast(u, v) \) exists and \( u' \equiv^\vee u \) and \( v' \equiv^\vee v \) hold. By Corollary II.4.36, we have \( u\theta^\ast(u, v) \equiv^\vee \theta^\ast(v, u) \), whence \( u\theta^\ast(u, v) \equiv^\vee v\theta^\ast(v, u) \). By assumption, the quadruple \( (u', v', \theta^\ast(u, v), \theta^\ast(v, u)) \) is \( \equiv \)-factorable: this means that \( \theta^\ast(u', v') \) and \( \theta^\ast(v', u') \) are defined and there exists \( w \) satisfying
\[
\theta^\ast(u, v) \equiv^\vee \theta^\ast(u', v') \text{ and } \theta^\ast(v, u) \equiv^\vee \theta^\ast(v', u').
\]
By a symmetric argument, we obtain the existence of \( w' \) satisfying
\[
\theta^\ast(u', v') \equiv^\vee \theta^\ast(u, v) w' \quad \text{and} \quad \theta^\ast(v', u') \equiv^\vee \theta^\ast(v, u) w'.
\]
Merging, we deduce \( \theta^\ast(u, v) \equiv^\vee \theta^\ast(u, v) w' w \). By Corollary II.4.45, the category \( (\mathcal{S} \mid \mathcal{R})^\ast \) is left-cancellative, so we deduce \( \varepsilon \equiv^\vee w' w \), whence \( w' = w = \varepsilon \) and, finally, \( \theta^\ast(u', v') \equiv^\vee \theta^\ast(u, v) \).

(ii) \( \Rightarrow \) (iii) Assume that \( \theta^\ast \) is compatible with \( \equiv^\vee \) and \( \theta^\ast_3(u, v, w) \) is defined. By Lemma II.4.6, we have \( \theta^\ast_3(u, v, w) = \theta^\ast(u(R\mathcal{C}^\ast(u, v), w)) \). As \( u(\theta^\ast(u, v)) \equiv^\vee v\theta^\ast(v, u) \) holds, the compatibility assumption implies that \( \theta^\ast(v(R\mathcal{C}^\ast(u, v), w)) \) is defined as well, and it is \( \equiv^\vee \)-equivalent to the latter. Now, by Lemma II.4.6 again, the path \( \theta^\ast(v(\theta^\ast(v, u), w)) = \theta^\ast_3(v, u, w) \), so we conclude that \( \theta^\ast_3(v, u, w) \) is defined and it is \( \equiv^\vee \)-equivalent to \( \theta^\ast_3(u, v, w) \), that is, the \( \theta \)-cube condition is true for \( (u, v, w) \).

(iii) \( \Rightarrow \) (i) Lemma II.4.61 gives the result directly.

Exercice 23 (alternative proof).— Assume that \( (\mathcal{S}, \mathcal{R}) \) is a right-complemented presentation associated with the syntactic right-complement \( \theta \), that \( (\mathcal{S}, \mathcal{R}) \) is right-Noetherian, and that the \( \theta \)-cube condition is true on \( \mathcal{S} \). (i) Show that, for all \( r, s, t \) in \( \mathcal{S} \), the path \( \theta(r, s\theta(s, t)) \) is defined if and only if \( \theta^\ast(r, t\theta(t, s)) \) is and, in this case, the relations \( \theta^\ast(r, s\theta(s, t)) \equiv^\vee \theta^\ast(r, t\theta(t, s)) \) and \( \theta^\ast(s\theta(s, t), r) \equiv^\vee \theta^\ast(t\theta(t, s), r) \) are satisfied. (ii) Show that the map \( \theta^\ast \) is compatible with \( \equiv^\vee \), that is, the conjunction of \( u' \equiv^\vee u \) and \( v' \equiv^\vee v \) implies that \( \theta^\ast(u', v') \) exists if and only if \( \theta^\ast(u, v) \) does and, in this case, they are \( \equiv^\vee \)-equivalent. [Hint: Show using on \( \lambda^\ast(u\theta^\ast(u, v)) \), where \( \lambda^\ast \) is a right-Noetherian witness for \( (\mathcal{S}, \mathcal{R}) \) that, if \( \theta^\ast(u, v) \) is defined and we have \( u' \equiv^\vee u \) and \( v' \equiv^\vee v \), then \( \theta^\ast(u', v') \) is defined and we have \( \theta^\ast(u', v') \equiv^\vee \theta^\ast(u, v) \).]
and \( \theta^*(v', u') \equiv_R^+ \theta^*(v, u) \).

(ii) The result of (i) is the compatibility in the case \( u' = u = r \) and \( v = s \theta(s, t) \), \( v' = \theta(t, s) \), that is, in the basic case of equivalence. We establish the general result using induction on \( \lambda^*(u \theta^*(u, v)) \) and, for a given value \( \alpha \), on the sum \( d \) of the combinatorial distances from \( u \) to \( u' \) and from \( v \) to \( v' \). For an induction, it is sufficient to consider the case \( d = 1 \), that is, we may assume \( u' = u \) and \( \text{dist}(v, v') = 1 \), that is, there exist \( s, t \) in \( S \) and \( v_0, v_1 \) in \( S^* \) satisfying \( v = v_0s \theta(s, t)v_1 \) and \( v' = v_0 \theta(t, s)v_1 \).

We assume that \( \theta^*(u, v) \) is defined, and our aim is to show that \( \theta^*(u', v') \) is defined as well and we have \( \theta^*(u', v') \equiv_R^+ \theta^*(u, v) \) and \( \theta^*(v', u') \equiv_R^+ \theta^*(v, u) \). To this end, we compare the reversing grids below:

![Diagram of reversing grids](image)

By assumption, the left grid exists, and we wish to show that the right grid exists as well and that the corresponding paths are pairwise \( \equiv_R^+ \)-equivalent. The rectangles on the left (\( u \) and \( v_0 \)) coincide. Next, we find a rectangle as in (i), namely \( r \) and \( s \theta(s, t) \) vs. \( r \) and \( \theta(t, s) \). By (i), \( u_3' \) and \( v_3' \) exist and we have \( u_3' \equiv_R^+ u_2 \) and \( v_3' \equiv_R^+ v_3 \). Then consider the median bottom rectangles (\( u_1 \) and \( v_1' \)): the point is the inequality

\[
\lambda^*(\theta^*(u_1v_5)) \leq \lambda^*(ru_1v_5) \leq \lambda^*(r_0u_1v_5) = \lambda^*(u, v) \leq \alpha.
\]

As \( \theta^*(u_1, v_3) \) exists and we have \( v_3' \equiv_R^+ v_3 \), the induction hypothesis implies that \( \theta^*(u_1, v_3') \) exists as well and gives \( u_4' \equiv_R^+ u_4 \) and \( v_5' \equiv_R^+ v_5 \). The argument is the same for the two right squares.

(iii) So, if the \( \theta \)-cube condition is true on \( S \), the map \( \theta^* \) is compatible with \( \equiv_R^+ \).

By Exercise 22, the latter condition implies that right-reversing is complete for \( (R, S) \).
Let $s$ from Proposition III.1.12, and more precisely from (III.1.13). Assume $s$ have the following conditions are equivalent:

- For $f,g,f',g'$ in a cancellative category $C$ satisfying $f'g = g'f$, the following conditions are equivalent:
  - (i) The elements $f'$ and $g'$ are left-disjoint;
  - (ii) The element $f'g$ is a weak left-lcm of $f$ and $g$.

Chapter III: Normal decompositions

Skipped proofs

Proposition III.1.14 (power).— If $S$ is a subfamily of a left-cancellative category $C$ and $g_1\cdot\ldots\cdot g_p$ is $S$-greedy, then $g_1\cdot\ldots\cdot g_m\mid g_{m+1}\cdot\ldots\cdot g_p$ is $S^m$-greedy for $1 \leq m \leq p$, that is,

\[(III.1.15) \quad \text{Each relation } s \leq f_1\cdot\ldots\cdot f_p \text{ with } s \in S^m \text{ implies } s \leq f_1\cdot\ldots\cdot f_m.\]

Proof. (See Figure 4.) We use induction on $m$. For $m = 1$, the result follows from Proposition III.1.12, and more precisely from (III.1.13). Assume $m \geq 2$. Let $s \in S^m$, say $s = s_1\cdot\ldots\cdot s_m$ with $s_1,\ldots, s_m$ in $S$, and $s \leq f_1\cdot\ldots\cdot f_p$. Then we have $s_1 \leq f_1\cdot\ldots\cdot f_p$, hence, by (III.1.13), $s_1 \leq f_1$, say $f_1 = s_1f_1$, and, therefore, $s_2\cdot\ldots\cdot s_m \leq f_1f_2\cdot\ldots\cdot f_p$. As $s_2\cdot\ldots\cdot s_m$ belongs to $S^{m-1}$, the induction hypothesis implies $s_2\cdot\ldots\cdot s_m \leq f_1f_2\cdot\ldots\cdot f_m$, whence $s_1s_2\cdot\ldots\cdot s_m \leq f_1g_2\cdot\ldots\cdot g_m$, as expected. $\square$

Lemma III.2.52.— For $f,g,f',g'$ in a cancellative category $C$ satisfying $f'g = g'f$, the following conditions are equivalent:

- (i) The elements $f'$ and $g'$ are left-disjoint;
- (ii) The element $f'g$ is a weak left-lcm of $f$ and $g$. 

Figure 3. Solution to Exercise 24: The cube condition viewed as a factorization property: whenever $\overline{uvwv}$ right-reverses to $\overline{vu}$, the quadruple $(u,v,u,v)$ is $\prec$-factorable.
Proof. Assume (i), and let assume that \( f''g = g''f \) is a common left-multiple of \( f \) and \( g \) such that \( f''g \) and \( f'g \) admit a common left-multiple, say \( h'(f''g) = h''(f'g) \). By assumption we also have \( h'(g''f) = h''(g'f) \) and, because \( \mathcal{C} \) is assumed to be right-cancellative, we deduce \( h'f''f' = h''f'f' \) and \( h'g''g' = h''g'g' \). Thus \((h', h'')\) witnesses for \((f'', g'') \bowtie (f', g')\). As \( f' \) and \( g' \) are left-disjoint, we deduce \( h' \preceq h'' \), that is, there exists \( h \) satisfying \( h'' = h'h \). We deduce \( h'(f''g) = h''(f'g) = h'h(f'g) \), whence \( f''g = h(f'g) \) by left-cancelling \( h' \). This shows that \( f''g \) is a left-multiple of \( f'g \), and the latter is a weak left-lcm of \( f \) and \( g \). So (i) implies (ii).

Assume now (ii), and let assume that \((h', h'')\) witnesses for \((f'', g'') \bowtie (f', g')\), that is, \( f'f'' = h'f'f' \) and \( h'g''g' = h''g'g' \).\( f'f''g = h''f'g = h''g'f = h'g''f' \), whence \( f''g = g''f \) by left-cancelling \( h' \). So \( f''g \) is a common left-multiple of \( f \) and \( g \). Moreover, the above equalities show that \( f''g \) and \( f'g \) admit a common left-multiple. As \( f''g \) is a weak left-lcm of \( f \) and \( g \), we deduce that \( f''g \) is a left-multiple of \( f'g \), that is, there exists \( h \) satisfying \( f''g = h'f'g \), hence \( h''f'g = h'f''g = h'hf'g \). Right-cancelling \( f'g \), we deduce \( h'' = h'h \), that is, \( h' \preceq h'' \). Hence \( f' \) and \( g' \) are left-disjoint, and (ii) implies (i).

Proposition III.2.53 (symmetric normal exist)—If \( \mathcal{S} \) is a Garside family in a cancellative category \( \mathcal{C} \), the following conditions are equivalent:

(i) For all \( f, g \) in \( \mathcal{C} \) admitting a common right-multiple, there exists a symmetric \( \mathcal{S} \)-normal path \( \overline{uv} \) satisfying \((f, g) \bowtie ([u]^+, [v]^+)\);

(ii) The category \( \mathcal{C} \) admits conditional weak left-lcms.

Proof. Assume (i), and let \( f, g \) be two elements of \( \mathcal{C} \) that admit a common left-multiple, say \( f\hat{g} = \hat{g}f \). By (i) applied to \( f \) and \( \hat{g} \), there exists a symmetric \( \mathcal{S} \)-normal path \( \overline{uv} \) such that, putting \( f' = [u]^+ \) and \( g' = [v]^+ \), we have \((f, \hat{g}) \bowtie (f', g')\), that is, there exist \( h', h \) satisfying \( h'f' = \hat{h}f' \) and \( h'\hat{g} = \hat{h}g' \). By Proposition III.2.11, \( f' \) and \( g' \) are left-disjoint, so, by Lemma III.2.52, \( f'g' \), which is also \( g'f' \), is a weak left-lcm of \( f \) and \( g \). Hence \( \mathcal{C} \) admits conditional weak left-lcms, and (i) implies (ii).

Conversely, assume (ii), and let \( f, g \) be two elements of \( \mathcal{C} \) that admit a common right-multiple, say \( f\hat{g} = \hat{g}f \). By (ii), there exists a weak left-lcm of \( f' \) and \( g' \), say \( f''g' = g''f' \), and \( h \) satisfying \( f''g' = h''f'g' \), hence \( f = hf'' \) by right-cancelling \( g' \) and, similarly, \( g = hg'' \). By Lemma III.2.52, the elements \( f''g' \) and \( g''f' \) are left-disjoint. Let \( u \) be an \( \mathcal{S} \)-normal decomposition of \( f''g' \) and \( v \) be an \( \mathcal{S} \)-normal decomposition of \( g''f' \). By (the trivial part of) Proposition III.2.11, the first entries of \( u \) and \( v \) are left-disjoint since \( f''g' \) and \( g''f' \) are, so \( \overline{uv} \) is symmetric \( \mathcal{S} \)-normal. Finally, the pair \((h, 1_x)\) (with \( x \) the source of \( f \) and \( g \)) witnesses for \((f, g) \bowtie (f'', g'')\). So (ii) implies (i).

Lemma III.2.55.—A cancellative category \( \mathcal{C} \) admits conditional weak left-lcms if and only if \( \mathcal{C} \) is a strong Garside family in itself.
Proof. Assume that $C$ is strong and $fs = gt$ holds. Then there exist $f', g', h$ such that $f'$ and $g'$ are left-disjoint and we have $f's = h't$, $f = h'f'$, and $g = hg'$. By Lemma III.2.52, $f'g$ is a weak left-lcm of $f$ and $g$, of which $fg$ is a left-multiple. So $C$ admits conditional weak left-lcms.

Conversely, assume that $C$ admits conditional weak left-lcms, and $fs = gt$ holds. Then there exists a weak left-lcm $f's$ of $s$ and $t$ of which $fs$ is a left-multiple. By Lemma III.2.52 again, $f'$ and $g'$ are left-disjoint, and the condition of Definition III.2.54 is satisfied. So $C$ is strong.

\[\text{Proposition III.2.56 (symmetric normal, short case III).— If } S \text{ is a strong Garside family in a cancellative category } C \text{ admitting conditional weak left-lcms, Algorithm III.2.42 running on a positive–negative } S^2\text{-path } v^n u^m \text{ such that } [u]^+ \text{ and } [v]^+ \text{ admit a common left-multiple, say } f[v]^+ = g[u]^+, \text{ returns a symmetric } S\text{-normal path } v^n u^m \text{ satisfying } (f, g) \succ (|u|^+, |v'^n|^+) \text{ and } [u'^n v]^+ = [v'^n u]^+, \text{ moreover there exists } h \text{ satisfying } f = h[u'^n]^+ \text{ and } g = h[v'^n]^+.\]

Once again, Proposition III.2.56 reduces to Proposition III.2.44 in the case of a left-Ore category as it then says that $f'$ is a weak left-lcm in $C$ that admits conditional weak left-lcms.

Proof. The argument is the same as for Proposition III.2.44, the only difference being that, at each left-reversing step, one has to check the existence of a factorization of the initial equality $f[v] = g[u]$. The principle is explained in Figure 5: the induction hypothesis that is maintained at each step in the construction of the rectangular diagram is that, for every local North-West corner in the current diagram, there exists a factorizing arrow coming from the top-left object $x$. When one more tile is added, the defining property of a strong Garside family guarantees that one can add a new tile in which the left and top arrows represent left-disjoint elements and there exists a factorizing arrow coming from $x$. The rest of the proof is unchanged as, in particular, the third domino rule is still valid in the extended context.

\[\text{Solution to selected exercises}\]

\[\text{Exercice 28 (invertible).— Assume that } C \text{ is a left-cancellative category and } S \text{ is included in } C. \text{ Show that, if } g_1 \cdots g_p \text{ belongs to } S^2, \text{ then } g_1 \cdots |g_p| \text{ being } S\text{-greedy implies that } g_2, \ldots, g_p \text{ are invertible.}\]

Solution. The element $g_1 \cdots g_p$ lies in $S^2$ is equal to, hence left-divides, itself. By Proposition III.1.12, $g_1 |g_2 \cdots g_p$ is $S$-greedy, so we deduce that $g_1 \cdots g_p$ left-divides $g_1$, say $g_1 = g_1 \cdots g_p g'$. Left-cancelling $g_1$, we deduce that $g_2 \cdots g_p g'$ is an identity-element, hence $g_2, \ldots, g_p$, and $g'$ must be invertible.

\[\text{Exercice 29 (deformation).— Assume that } C \text{ is a left-cancellative category. Show that a path } g_1 \cdots |g_p| \text{ is a } C^\epsilon\text{-deformation of } f_1 \cdots |f_p| \text{ if and only if } g_1 \cdots g_i =^* f_1 \cdots f_i \text{ holds for } 1 \leq i \leq \max(p, q), \text{ the shorter path being extended by identity-elements if needed.}\]

Solution. Let $r = \max(p, q)$. Assume that $\epsilon_0, \ldots, \epsilon_r$ are invertible elements witnessing that $g_1 \cdots |g_r|$ is a $C^\epsilon$-deformation of $f_1 \cdots |f_r|$. For every $i$, we deduce $f_1 \cdots f_i \epsilon_i = \epsilon_0 g_1 \cdots g_r$, whence $g_1 \cdots g_i =^* f_1 \cdots f_i$ since $\epsilon_0$ is an identity-element.
Addenda to “Foundations of Garside Theory”, by Dehornoy et al.

Conversely, assume \( g_1 \cdots g_i = f_1 \cdots f_i \) for every \( i \), say \( g_1 \cdots g_i = f_1 \cdots f_i \epsilon_i \) with \( \epsilon_i \) in \( \mathbf{C} \). Set \( \epsilon_0 = 1_x \) where \( x \) is the source of \( f_1 \). Then we have \( f_1 \epsilon_1 = g_1 \) by construction. Assume \( i \geq 2 \). Then we obtain \( f_1 \cdots f_{i-1} f_i \epsilon_i = g_1 \cdots g_{i-1} g_i = f_1 \cdots f_{i-1} \epsilon_{i-1} g_i \), whence \( f_i \epsilon_i = \epsilon_{i-1} g_i \) by left-cancelling \( f_1 \cdots f_{i-1} \). So \( g_1 | \cdots | g_r \) is a \( \mathbf{C} \times \)-deformation of \( f_1 | \cdots | f_r \).

**Exercice 33 (left-disjoint).**— Assume that \( \mathbf{C} \) is a left-cancellative category, \( f \) and \( g \) are left-disjoint elements of \( \mathbf{C} \), and \( f \) left-divides \( g \). Show that \( f \) is invertible.

**Solution.** Let \( x \) be the common source of \( f \) and \( g \). By assumption we have \( f \preceq 1_x f \) and \( f \preceq 1_x g \). By definition of \( f \) and \( g \) being left-disjoint, this implies \( f \preceq 1_x \), hence \( f \) must be invertible.

**Exercice 34 (normal decomposition).**— Give a direct argument from deriving Proposition III.2.20 from Corollary III.2.50 in the case when \( S \) is strong.

**Solution.** Let \( gf^{-1} \) be an element of \( \mathbf{C} \mathbf{C}^{-1} \). Let \( s_1 | \cdots | s_p \) be an \( S \)-normal decomposition of \( f \), and let \( t_1 | \cdots | t_q \) be an \( S \)-normal decomposition of \( g \). We prove the existence of an \( S \)-normal decomposition for \( gf^{-1} \) using an induction on \( p \) to construct a rectangular diagram consisting of \( p \) rows of \( q \) tiles as in Lemma III.2.31. As \( t_1 | \cdots | t_q \) is \( S \)-normal, Corollary III.2.50 inductively implies that the elements of every horizontal line of the diagram make an \( S \)-normal path, and so do in particular the elements \( t_1' | \cdots | t_q' \) of the top line. Similarly, as \( s_1 | \cdots | s_p \) is \( S \)-normal, Corollary III.2.50 again inductively implies that the elements of every vertical line of the diagram make an \( S \)-normal path, and so do in particular the elements \( s_1' | \cdots | s_p' \) of the left line. Finally, \( s_1' \) and \( t_1' \) are left-disjoint by construction. Hence \( s_1' | \cdots | s_p' | t_1' | \cdots | t_q' \) is an \( S \)-normal decomposition of \( gf^{-1} \).

**Exercice 35 (Garside base).**— (i) Let \( \mathcal{G} \) be the category whose diagram is displayed on Figure 6 left, and let \( S = \{ a, b \} \). Show that \( \mathcal{G} \) is a groupoid with nine elements, \( S \) is a Garside base in \( \mathcal{G} \), the subcategory \( \mathbf{C} \) of \( \mathcal{G} \) generated by \( S \) contains

![Diagram of Algorithm III.1.52](image-url)
no nontrivial invertible element, but $\mathcal{C}$ is not an Ore category. Conclusion? (ii) Let $\mathcal{G}$ be the category whose diagram is displayed on Figure 6 right, let $\mathcal{S} = \{\epsilon, a\}$, and let $\mathcal{C}$ be the subcategory of $\mathcal{G}$ generated by $\mathcal{S}$. Show that $\mathcal{G}$ is a groupoid and every element of $\mathcal{G}$ admits a decomposition that is symmetric $\mathcal{S}$-normal in $\mathcal{C}$. Show that $\epsilon a$ admits a symmetric $\mathcal{S}$-normal decomposition and no $\mathcal{S}$-normal decomposition. Is $\mathcal{S}$ a Garside family in $\mathcal{C}$? Conclusion?

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Diagrams of the categories of Exercise 35.}
\end{figure}

\textbf{Solution.} (i) The nine elements of $\mathcal{G}$ are $1_0$, $1_1$, $1_2$, $a$, $b$, $a^{-1}$, $b^{-1}$, $a^{-1}b$, and $b^{-1}a$. That $\mathcal{S}$ is a Garside base in $\mathcal{G}$ follows from a direct inspection. The subcategory of $\mathcal{G}$ generated by $\mathcal{S}$ comprises five elements: $1_0$, $1_1$, $1_2$, and $a, b$, none of which is invertible. Next, $\mathcal{C}$ is not an Ore category since the elements $a$ and $b$ have non common right-multiple although they share the same source. So, in Proposition III.2.25, the conclusion cannot be strengthened to claim that $\mathcal{C}$ necessarily is an Ore category (whereas the assumption of Proposition III.2.24 cannot be weakened to only assume that $\mathcal{C}$ is a left-Ore category).

(ii) The elements of $\mathcal{G}$ are $1_0$, $1_1$, $\epsilon$, $a$, $\epsilon a$, $a^{-1}$, and $a^{-1} \epsilon$; those of $\mathcal{C}$ are $1_0$, $1_1$, $\epsilon$, $a$, $\epsilon a$. Then $\epsilon a$ admits the symmetric $\mathcal{S}$-normal decomposition $\epsilon a$, but admits no $\mathcal{S}$-normal decomposition. Hence $\mathcal{S}$ is not a Garside family in $\mathcal{C}$. So, in Proposition III.2.25, we cannot simply drop the assumption about nontrivial invertible elements.

Chapter IV: Recognizing Garside families

\textbf{Skipped proofs}

Lemma IV.1.13.— Assume that $\mathcal{S}$ is a subfamily of a left-cancellative category $\mathcal{C}$.

(ii) The family $\mathcal{S}$ is closed under right-comultiple if and only if $\mathcal{S}^\dagger$ is.

(iii) If $\mathcal{S}$ is closed under right-complement and $\mathcal{C}^\dagger \mathcal{S} \subseteq \mathcal{S}$ holds, then $\mathcal{S}^\dagger$ is closed under right-complement too.

Proof. (ii) Assume that $\mathcal{S}^\dagger$ is closed under right-comultiple, and we have $sg = tf$ with $s, t$ in $\mathcal{S}$. The assumption implies the existence of $s', t', h$ satisfying $st' = ts' \in \mathcal{S}^\dagger$, $f = s'h$, and $g = t'h$. If $st'$ is invertible, then $s$ and $t$ must be invertible too, so $s$ is a common right-multiple of $s$ and $t$ lying in $\mathcal{S}$ of which $sg$ is a right-multiple. Assume now that $st'$ is not invertible. Then there exists $\epsilon$ in $\mathcal{C}$ such that $st' \epsilon^{-1}$ lies in $\mathcal{S}$. Then we have $s(t' \epsilon^{-1}) = s(t' \epsilon^{-1}) \in \mathcal{S}$, $f = (s' \epsilon^{-1})(ch)$, and $g = (t' \epsilon^{-1})(eh)$. Hence $s' \epsilon^{-1}$, $t' \epsilon^{-1}$, and $eh$ witness for $\mathcal{S}$ being closed under right-comultiple.

Conversely, assume that $\mathcal{S}$ is closed under right-comultiple and we have $sg = tf$ with $s, t \in \mathcal{S}^\dagger$. Assume first that $s$ or $t$ is invertible, say $s$. Put $s' = 1_y$ where $y$ is
the target of $t$, $t' = s^{-1}t$, and $h = f$. Then we have $st' = ts'$, $f = s'h$, $g = t'h$, and $s'$ and $t'$ are invertible, hence lie in $S^f$, and, by assumption, so does $st'$, which is $t$. So $st'$ is a common right-multiple of $s$ and $t$ that lies in $S^f$ and of which $sg$ is a right-multiple. Assume now that neither $s$ nor $t$ is invertible. Then $s$ and $t$ lie in $SC^s$, and there exist $s', t'$ in $S$ and $s, t'$ in $C^s$ satisfying $s = s'\epsilon$ and $t = t'\epsilon'$, see Figure 7. Then we have $s'(eg) = t'(f'g)$ with $s', t'$ in $S$. As $S$ is closed under right-comultiple, there must exist $s'', t''$, and $h$ satisfying

$$s't'' = t's'' \in S, \quad \epsilon'f = s''h, \quad \text{and} \quad eg = t''h.$$ 

As $\epsilon$ and $\epsilon'$ are invertible, we can put $s'' = \epsilon^{-1}s''$ and $t'' = \epsilon^{-1}t''$. Then we have $f = s''h$ and $g = t''h$, and $st'' = st'' = t's'' = ts'' \in S \subseteq S^f$. So, again, $st''$ is a common right-multiple of $s$ and $t$ that lies in $S^f$ and of which $sg$ is a right-multiple. Hence $S^f$ is closed under right-comultiple.

(iii) Assume now that $S$ is closed under right-complement, $SC^s \subseteq S$ holds, and we have $s, t \in S^f$ and $sg = tf$. We follow the same scheme as for (ii), and keep the same notation. Assume first that $s$ or $t$ is invertible, say $s$. Put $s'' = 1_y$ (the target of $t$, $t'' = s^{-1}t$, and $h = f$. Then we have $st'' = ts''$, $f = s''h$, and $g = t''h$. Moreover, $s''$ belongs to $S^f$ by definition and $t'$, which belongs to $C^sS^f$, hence to $C^s \cup C^sSC^s$, belongs to $S^f$ as $C^sS$ is included in $S$. Assume now that neither $s$ nor $t$ is invertible. Then we write $s = s'\epsilon$ and $t = t'\epsilon'$ with $s', t'$ in $S$ and $\epsilon, \epsilon'$ in $C^s$, see Figure 7 again. We have $s'(eg) = t'(f'g)$ so, as $S$ is closed under right-complement, there exist $s'', t''$ in $S$ and $h$ in $C$ satisfying $s't'' = ts''$, $eg = t''h$, and $\epsilon'f = s''h$. Put $s'' = \epsilon^{-1}s''$ and $t'' = \epsilon^{-1}t''$. Then we have $st'' = ts''$, $f = s''h$, and $g = t''h$. Moreover, by construction, $s''$ and $t''$ belong to $C^sS$, hence, by assumption, to $S^f$. Hence $S^f$ is closed under right-complement.

\[\begin{align*}
  s' & \xleftarrow{f} t' \\
  s & \xleftrightarrow{s''h} f \\
  g & \xrightarrow{t''h} h
\end{align*}\]

**Figure 7. Proof of Lemma IV.1.13.**

**Lemma IV.2.39.**— For every subfamily $S$ of a left-cancellative category $C$ that is right-Noetherian and admits unique conditional right-lcms, the following conditions are equivalent:

(i) The family $S^f$ is closed under right-complement (in the sense of Definition IV.1.3);

(ii) The family $S^f$ is closed under $\setminus$, that is, if $s$ and $t$ belong to $S^f$, then so does $s \setminus t$ when defined, that is, when $s$ and $t$ admit a common right-multiple.

**Proof.** Assume that $S^f$ is closed under right-complement. Let $s, t$ be elements of $S^f$ that admit a common right-multiple, and let $h$ be the right-lcm of $s$ and $t$. By
assumption, we have \( h = s(t) = t(s) \). As \( S' \) is closed under right-complement, there exist \( s', t' \in S' \) and \( h' \) satisfying \( st' = ts', t's = s'h' \), and \( s' t = t'h' \), whence \( h = (st') h' \). By definition of the right-lcm, \( h' \) must be an identity-element and, therefore, \( s't \) and \( t's \) belong to \( S' \). So \( S' \) is closed under the right-complement operation, and (i) implies (ii).

Conversely, assume that \( S' \) is closed under the right-complement operation. Assume that \( s, t \) belong to \( S' \) and \( sg = tf \) holds. Then \( sg \) is a common right-multiple of \( s \) and \( t \), hence it is a right-multiple of their right-lcm, which is \( s(t) \) and \( t(s) \). So there exists \( h \) satisfying \( sg = s(t) h \). Left-cancelling \( s \), we deduce \( g = (s \setminus t) h \) and, symmetrically, \( f = (t \setminus s) h \). Then \( t \setminus s, s \setminus t, \) and \( h \) witness that the expected instance of closure under right-complement is satisfied. So (ii) implies (i).

\[ \square \]

Solution to selected exercises

**Exercise 38 (multiplication by invertible).—** Assume that \( C \) is a cancellative category, and \( S \) is a subfamily of \( C \) that is closed under left-divisor and contains at least one element with source \( x \) for each object \( x \). Prove \( S^2 = S \).

**Solution.** Assume \( g \in S \) and \( e \in C \). Then we have \( g = g e \epsilon^{-1} \), whence \( ge \preceq g \), and \( ge \in S \). So \( S^2 \) is included in \( S \).

**Exercise 40 (head vs. lcm).—** Assume that \( C \) is a left-cancellative category, \( S \) is included in \( C \), and \( g \) belongs to \( C \setminus C' \). Show that \( s \) is an \( S \)-head of \( g \) if and only if it is a right-lcm of \( \text{Div}(g) \cap S \).

**Solution.** If \( s \) is an \( S \)-head of \( g \), then there exists \( g' \) so that \( s | g' \) is an \( S \)-cancellation decomposition of \( g \). By Lemma IV.1.21, we deduce that, for \( t \in S \), the relation \( t \preceq g \) implies \( t \preceq s \); this means that every element of \( \text{Div}(g) \cap S \) divides \( s \) and, therefore, that \( s \) is a left-lcm of \( \text{Div}(g) \cap S \).

**Exercise 41 (closed under right-comultiple).—** Assume that \( C \) is a cancellative category, \( S \) is a subfamily of \( C \), and there exists \( H : C \setminus C' \to S \) satisfying (IV.1.46). Show that \( S \) is closed under right-comultiple.

**Solution.** Assume \( f, g \in S \) and \( fg = gf \). If \( f \circ g \) is invertible, then everything is obvious. Otherwise, \( H(fg) \) is defined, and, by (IV.1.46)(i), we have \( H(fg) \preceq fg \). On the other hand, we have \( f \in S \) and \( f \preceq fg \), whence \( f \preceq H(fg) \) by (IV.1.46)(iii). A symmetric argument gives \( g \preceq H(fg) \) as we have \( fg = gf \). So \( H(fg) \), which belongs to \( \text{S} \) by construction, is a common right-multiple of \( f \) and \( g \) of the expected type.

**Exercise 42 (power).—** Assume that \( S \) is a Garside family a left-cancellative category \( C \). Show that, if \( g_1 | \cdots | g_p \) is an \( S^m \)-normal decomposition of \( g \) and, for every \( i \), the path \( s_{1,i} | \cdots | s_{i,m} \) is an \( S \)-normal decomposition of \( g_i \), then the path \( s_{1,1} | \cdots | s_{m,1} | \cdots | s_{2,1} | \cdots | s_{2,m} | \cdots | s_{p,1} | \cdots | s_{p,m} \) is an \( S \)-normal decomposition of \( g \).

**Solution.** By assumption, every \( s_{i,j} \) lies in \( S \) and \( s_{i,j} | s_{i,j+1} \) is \( S \)-greedy for all \( i, j \). So the point is to show that \( s_{i,m} | s_{i+1,1} \) is greedy. Now assume \( t \in s_{l,m} s_{l+1,1} \) with \( t \) in \( S \). Then we deduce \( s_{l,1} | \cdots | s_{l,m} | s_{l+1,1} \) is \( S \)-greedy, whence \( s_{l,1} | \cdots | s_{l,m} | s_{l+1,1} \) is \( S \)-greedy. As \( g_1 g_2 \) is \( S^m \)-greedy, we deduce \( s_{l,1} | \cdots | s_{l,m} | t \preceq g_1 \), that is, \( s_{l,1} | \cdots | s_{l,m} | t \preceq s_{l,1} | \cdots | s_{l,m} | s_{l+1,1} \), which implies \( t \preceq s_{l+1,1} \) since \( g \) is left-cancellative. As \( S \) is a Garside family, Corollary IV.1.31 implies that this is enough to conclude that \( s_{l,m} | s_{l+1,1} \) is \( S \)-greedy.
**Exercice 45 (no conditional right-lcm).**— Let $M$ be the monoid generated by $a, b, a', b'$ subject to the relations $ab = ba, a'b' = b'a', aa' = bb', a'a = b'b$. (i) Show that the cube condition is satisfied on $\{a, b, a', b'\}$, and that right- and left-reversing are complete for the above presentation. (ii) Show that $M$ is cancellative and admits right-mcms. (iii) Show that $a$ and $b$ admit two right-mcms in $M$, but they admit no right-lcm. (iv) Let $S = \{a, b, a', b', ab, a'b', aa', a'a\}$, see diagram on the side. Show that $S$ is closed under right-mcm, and deduce that $S$ is a Garside family in $M$. (v) Show that the (unique) strict $S$-normal decomposition of the element $a^2 b' a'^2$ is $ab|a'b'|b'$.

![Diagram](image)

**Solution.** (i) There are several relations of the form $a \cdots = b \cdots$, but, as there is no relation $a \cdots = a' \cdots$, there is no mixed cube to consider: the only triples to check are $(a, a, b)$ and the like, and this is easy.

(ii) The presentation is homogeneous, hence (strongly) Noetherian. By Lemma II.4.62, right-reversing is complete. Hence $M$ is right-cancellative and admits right-mcms. By symmetry of the relations, left-reversing is complete as well, and $M$ is right-cancellative.

(iii) By assumption, $ab$ and $aa'$ are common right-multiples of $a$ and $b$. Owing to their length, they must be minimal. As right-reversing is complete, every common right-multiple of $a$ and $b$ is a right-multiple or $ab$ and $aa'$. Hence the latter are the only mcms of $a$ and $b$. Finally neither is a right-multiple of the other, so $a$ and $b$ admit no right-lcm.

(iv) The set $S$ generates $M$, and it is closed under right-divisor and right-mcm: $ab$ and $aa'$ are common right-multiples of $a$ and $b$, $a'b'$ and $a'a$ are the two right-mcms of $a'$ and $b'$, whereas none of the pairs $\{a, a'\}, \{a, b'\}, \{b, a'\}$ admits a common right-multiple. Hence, by Corollary IV.2.26, $S$ is a Garside family in $M$.

(v) Push the letters to the left as much as possible.

**Exercice 47 (solid).**— Assume that $C$ is a left-cancellative category and $S$ is a generating subfamily of $C$. (i) Show that $S$ is solid in $C$ if and only if $S$ includes $1_C$ and it is closed under right-quotient. (ii) Assume moreover that $S$ is closed under right-divisor. Show that $S$ includes $C^\epsilon \setminus 1_C$, that $\epsilon \in S \cap C^\epsilon$ implies $\epsilon^{-1} \in S$, and that and $C^S = S$ holds, but that $S$ need not be solid.

**Solution.** (ii) Assume that $\epsilon$ is a nontrivial invertible element, with target $y$. As $S$ generates $C$, there exists at least one element $\epsilon'$ of $S$ that right-divides $\epsilon$. Then $1_y$ right-divides $\epsilon'$, which lies in $S$, hence $1_y$ must lie in $S$. Next, $\epsilon$ right-divides $1_y$, which lies in $S$, hence $\epsilon$ must lie in $S$. Now assume $\epsilon \in S \cap C^\epsilon$. Then either $\epsilon$ is an identity-element, in which case it coincides with $\epsilon^{-1}$ and the latter lies in $S$, or $\epsilon^{-1}$ is invertible and is not an identity-element, in which case it belongs to $S$ by the above argument. For $C^S = S$, the proof is as for Lemma IV.2.2. If $y$ is an object
Exercice 48 (solid).— Let $M$ be the monoid $(a,e | ea = a, e^2 = 1)^+$. (i) Show that every element of $M$ has a unique expression of the form $a^pe^q$ with $p \geq 0$ and $q \in \{0,1\}$, and that $M$ is left-cancellative. (ii) Let $S = \{1,a,e\}$. Show that $S$ is a solid Garside family in $M$, but that $S = S^2$ does not hold.

Solution. (i) Every word in $\{a,e\}^+$ is equivalent to a word of the form $a^pe^q$. Conversely, it is impossible that two distinct words of this form are equivalent, as the number of $a$ is an invariant, and so is the parity of the number of $e$ that follows the last $a$. The formulas $a \cdot a^pe^q = a^{p+1}e^q$ and $e \cdot a^pe^q = a^pe^q$ for $p \geq 1$, plus $e \cdot e^q = e^{q+1} \mod 2$ show that, for every $s$ in $\{a,e\}$, the value of $g$ can be recovered from $s$ and $qg$. (ii) $S$ generates $M$. The explicit formula for the multiplication shows that $a$ is right-divisible only by 1 and itself, and so does $e$. Hence $S$ is closed under right-divisor, and it is solid. For $p \geq 1$, define $H(a^pe^q) = a$. Then $H$ is a $S$-head function. Hence, by Proposition IV.2.7(i), $S$ is a Garside family in $M$. On the other hand, $ae$ is an element of $S^2 \setminus S$.

Exercice 49 (not solid).— Let $M = (a,e | ea = ae, e^2 = 1)^+$, and $S = \{a,e\}$. (i) Show that $M$ is left-cancellative. [Hint: $M$ is $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$] (ii) Show that $S$ is a Garside family in $M$, but $S$ is not solid in $M$. [Hint: $ea$ right-divides $a$, but does not belong to $S$.]

Solution. (i) Every word of $\{a,e\}^+$ is equivalent modulo the relations to a word of the form $a^pe^q$ with $q \leq 1$. As $(1,\bar{0})$ and $(0,\bar{1})$ satisfy in $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$ the defining relations of $M$, there exists a well defined homomorphism $F$ of $M$ to $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$ that satisfies $F(a) = (1,\bar{0})$ and $F(e) = (0,\bar{1})$. As $(1,\bar{0})$ and $(0,\bar{1})$ generate $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$, the homomorphism $F$ is surjective. As the images under $F$ of pairwise distinct words $a^pe^q$ are distinct, $F$ is injective, hence it is an isomorphism. Hence $\mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$ is left-cancellative, so is $M$. (ii) By definition, $S$ generates $M$. Next, the invertible elements of $M$ are 1 and $e$. The equality $ea = ae$ implies $M \cdot S \subseteq S^2$. Finally, the three elements of $S^2$, namely $a^2, ae$, and 1, admit $S$-normal decompositions, namely for instance $a|a,a|e$, and $e$. So $S$ is a Garside family in $M$. Now $ae$ does not lie in $S$, but we have $a = e(ae)$, so $S$ is not closed under right-divisor, hence is not solid.

Exercice 50 (recognizing Garside, right-lcm solid case).— Assume that $S$ is a solid subfamily in a left-cancellative category $C$ that is right-Noetherian and admits conditional right-lcms. Show that $S$ is a Garside family in $C$ if and only if $S$ generates $C$ and it is weakly closed under right-lcm.

Solution. If $S$ is a (solid or not solid) Garside family in $C$, then, by Corollary IV.2.29, $S$ is weakly closed under right-lcm. Conversely, assume that $S$ is solid, generates $C$, and is weakly closed under right-lcm. By definition, $S$ is closed under right-divisor, hence, by Lemma IV.1.13(i), so is $S^2$. Then Corollary IV.2.29(iii) implies that $S$ is a Garside family in $C$.

Exercice 52 (local right-divisibility).— Assume that $C$ is a left-cancellative category and $S$ is a generating subfamily of $C$ that is closed under right-divisor. (i) Show that the transitive closure of $\preceq_S$ is the restriction of $\preceq$ to $S$. (ii) Show that
the transitive closure of \( \preceq_{S} \) is almost the restriction of \( \preceq \) to \( S \), in the following sense: if \( s \succeq t \) holds, there exists \( s' = s \) satisfying \( s' \preceq_{S} t \).

**Solution.** Let \( \preceq_{S} \) be the transitive closure of \( \preceq_{S} \). As \( s \preceq_{S} t \) implies \( s \preceq t \) trivially and \( \preceq \) is transitive, \( s \preceq_{S} t \) implies \( s \preceq t \). Conversely, assume \( s \preceq t \), say \( t = gs \). As \( S \) generates \( C \), there exist \( s_{1}, \ldots, s_{q} \in S \) satisfying \( g = s_{1} \cdots s_{q} \). So we have \( t = s_{1} \cdots s_{p} \), \( s_{i} \in S \). Each element \( t_{i} \), right-divides \( t \), an element of \( S \), so it belongs to \( S \). Now, by construction, we have \( s = \tilde{s}_{G}t_{1} \tilde{s}_{G} \cdots \tilde{s}_{G}t_{p} = t \), whence \( s \preceq_{S} t \).

(ii) Let \( \preceq_{S} \) be the transitive closure of \( \preceq_{S} \). By Lemma IV.2.15, \( s \preceq_{S} t \) implies \( s \succeq t \), hence \( s \preceq_{S} t \) implies \( s \preceq t \) since \( \preceq \) is transitive. Conversely, assume \( s \preceq t \), say \( t = gs \) with \( g \notin C \). As \( S \) generates \( C \), there exist \( s_{1}, \ldots, s_{q} \in S \) satisfying \( g = s_{1} \cdots s_{q} \). Assume that \( p \) has been chosen minimal. Then \( s_{1}, \ldots, s_{p-1} \) are not invertible: if \( s_{i} \) is invertible, then \( s_{i}, s_{i+1} \) right-divides \( s_{i+1} \), hence belongs to \( S \), and therefore grouping \( s_{i} \) and \( s_{i+1} \) would provide a shorter decomposition. We have \( t = s_{1} \cdots s_{p} \). Put \( t_{i} = s_{1} \cdots s_{p} \) for \( 1 \leq i \leq p \). Each element \( t_{i} \), right-divides \( t \), an element of \( S \), so it belongs to \( S \). As \( S \) is closed under right-divisor, we have \( C^{s} S \subseteq S \). So we can assume that \( s_{1}, \ldots, s_{p-1} \) are non-invertible. Hence we have \( s_{1} \cdots s_{p} \).

**Exercise 53 (local left-divisibility).—** Assume that \( S \) is a subfamily of a left-cancellative category \( C \). (i) Show that \( s \preceq_{S} t \) implies \( s \preceq t \), and that \( s \preceq_{S} t \) is equivalent to \( s \preceq t \) whenever \( S \) is closed under right-quotient in \( C \). (ii) Show that, if \( S^{e} = C^{e} \cap S \) holds, then \( s \preceq_{S} t \) implies \( s \preceq t \). (iii) Show that, if \( S \) is closed under right-divisor, then \( s \preceq_{S} t \) is the restriction of \( \preceq \) to \( S \) and, if \( S^{e} = C^{e} \cap S \) holds, \( \preceq_{S} \) is the restriction of \( \preceq \) to \( S \).

**Solution.** (iii) First \( s \preceq_{S} t \) implies \( s \preceq t \). Conversely, assume \( s, t \in S \) and \( s \preceq t \). As \( t \) belongs to \( S \), the assumption that \( S \) is closed under right-divisor implies that \( g^{t} \) belongs to \( S \), hence witnesses for \( s \preceq_{S} t \). Next, by Lemma IV.2.15, \( s \preceq_{S} t \) implies \( s \preceq t \). Conversely, assume \( s \preceq_{S} t \) with \( t' \notin C^{e} \). As above, \( t' \) must belong to \( S \), and it cannot belong to \( S^{e} \). So \( s \preceq_{S} t \) holds.

**Exercise 55 (locally right-Noetherian).—** Assume that \( C \) is a left-cancellative category and \( S \) is a subfamily of \( C \). (i) Prove that \( S \) is locally right-Noetherian if and only if, for every \( s \in S^{e} \), every \( \preceq_{S} \)-increasing sequence in \( \text{Div}_{S}(s) \) is finite. (ii) Assume that \( S \) is locally right-Noetherian and closed under right-divisor. Show that \( S^{e} \) is locally right-Noetherian. [Hint: For \( X \subseteq S^{e} \) introduce \( X' = \{ s \in S \mid \exists \epsilon, \epsilon' \in C^{e} (e \epsilon' \in X') \} \), and construct a \( \preceq \)-minimal element in \( X \) from a \( \preceq \)-minimal element in \( X' \).]

**Solution.** (ii) Let \( X \) be a nonempty subfamily of \( S^{e} \). Put

\[
X' = \{ s \in S \mid \exists \epsilon, \epsilon' \in C^{e} (e \epsilon' \in X') \}.
\]

We have \( X \subseteq X' \), whence \( X' \neq \emptyset \). As \( X' \) is included in \( S \), it contains a \( \preceq_{S} \)-minimal element, say \( s_{0} \). By definition, there exists \( e_{0}, e_{0}' \) invertible such that \( e_{0}s_{0}e_{0}' \) lies in \( X \). Put \( t_{0} = e_{0}s_{0}e_{0}' \). We claim that \( t_{0} \) is \( \preceq_{S} \)-minimal in \( X \). Indeed, assume \( t_{0} = rt \) with \( r \in S^{e} \) and \( t \in X \). We have to prove that \( r \) lies in \( (S^{e})^{X} \). As \( S^{e} \) is solid, it suffices to show that \( r \) lies in \( C^{e} \). By definition, \( r \) belongs to \( SC^{e} \cup C^{e} \). If \( r \) belongs to \( C^{e} \), we are done. Otherwise, write \( r = r' \epsilon \) with \( r' \in S \) and \( \epsilon \in C^{e} \). Then we have \( s_{0} = (r'^{-1} \epsilon t)(\epsilon t^{-1}) \). As \( s_{0} \) lies in \( S \) and \( S \) is closed under right-divisor,
Chapter V: Bounded Garside families

Skipped proofs

Proposition V.1.59 (right-cancellative II).— If $S$ is a Garside family in a left-cancellative category $C$ that is right-bounded by a target-injective map $\Delta$, and $\phi_\Delta$ preserves $S$-normality and is surjective on $C^x$, then the following conditions are equivalent:

(i) The category $C$ is right-cancellative;
(ii) The functor $\phi_\Delta$ is injective on $C$;
(iii) The functor $\phi_\Delta$ is injective on $S^t$.

Proof. By Proposition V.1.36, (i) and (ii) are equivalent, and (ii) obviously implies (iii).

Now assume that $\phi_\Delta$ is injective on $S^t$. We will prove that it is injective on $C$. First, we claim that $\phi_\Delta(s') = \phi_\Delta(s)$ implies $s = s'$ for $s, s'$ in $S^t$. Indeed, assume $\phi_\Delta(s') = \phi_\Delta(s) \epsilon$ with $\epsilon \in C^e$. If $\epsilon$ is trivial, the injectivity of $\phi_\Delta$ on $S^t$ implies $s' = s$. Otherwise, as $\phi_\Delta$ is surjective on $C^e$, there exists $e'$ in $C^e$ satisfying $\phi_\Delta(e') = \epsilon$. As $\Delta$ is target-injective, the assumption that $\phi_\Delta(s)\phi_\Delta(e')$ is defined implies that $s'\phi_\Delta$ is defined too: if $y$ is the target of $s$ and $x'$ is the source of $e'$, we obtain $\phi_\Delta(y) = \phi_\Delta(x')$, whence $y = x'$ as $\phi_\Delta$ is injective on objects. We deduce $\phi_\Delta(s') = \phi_\Delta(s')\phi_\Delta$.

As $s$ belongs to $S^t$, so does $s'$, and the assumption that $\phi_\Delta$ is injective on $S^t$ then implies $s' = s'$, hence $s' = s$.

Now we prove using induction on $p \geq 1$ the statement: $\phi_\Delta(g) = \phi_\Delta(g')$ implies $g = g'$ for all $g, g'$ satisfying $\max(||g||_S, ||g'||_S) \leq p$. For $p = 1$, as $||g||_S \leq 1$ implies $g \in S^t$, the result follows from the assumption that $\phi_\Delta$ is injective on $S^t$. Assume $p \geq 2$, and $\phi_\Delta(g) = \phi_\Delta(g')$ with $\max(||g||_S, ||g'||_S) \leq p$. Let $s_1 \ldots s_p$ and $s_1' \ldots s_p'$ be $S$-normal decompositions of $g$ and $g'$. As $\phi_\Delta$ is a functor and it preserves normality, $\phi_\Delta(s_1) \cdots \phi_\Delta(s_p)$ and $\phi_\Delta(s_1') \cdots \phi_\Delta(s_p')$ are $S$-normal decompositions of $\phi_\Delta(g)$ and $\phi_\Delta(g')$. By Proposition III.1.25 (normal unique), $\phi_\Delta(g) = \phi_\Delta(g')$ implies $\phi_\Delta(s_1) = \phi_\Delta(s_1')$, whence, by the claim above, $s_1' = s_1$. Hence $s_1$ is an $S$-head for $g$ and $g'$, and we can write $g = s_1g_1$, $g' = s_1g_1'$, with $\max(||g_1||_S, ||g_1'||_S) \leq p - 1$. Then $\phi_\Delta(g) = \phi_\Delta(g)$ implies $\phi_\Delta(s_1)\phi_\Delta(g_1) = \phi_\Delta(s_1)\phi_\Delta(g_1')$, whence $\phi_\Delta(g_1) = \phi_\Delta(g_1')$. The induction hypothesis implies $g_1 = g_1'$, whence $g' = g$. So $\phi_\Delta$ is injective on $C$, and (iii) implies (ii).

Proposition V.2.34 (right-cancellativity III).— If $C$ is a left-cancellative category and $\Delta$ is a Garside map of $C$ that preserves $\Delta$-normality, then $C$ is right-cancellative if and only if $\phi_\Delta$ is injective on $(\text{Div}(\Delta))^2$. 

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te^{-1}_{t_0} \in S$. As $t$ belongs to $X$, we deduce that $te^{-1}_{t_0}$ lies in $X'$. Then, by the choice of $s_0$, the elements $e_{t_0}^{-1}r'$, and therefore $r'$ and $r\epsilon$, that is, $r$, must be invertible in $C$. Hence $t_0$ is $\prec_{S^t}$-minimal in $X$, and $\prec_{S^t}$ is a well-founded relation, that is, $S^t$ is locally right-Noetherian.
Proof. If $\mathcal{C}$ is right-cancellative, Proposition V.1.36 implies that $\phi_\Delta$ is injective on $\mathcal{C}$, hence a fortiori on $(\text{Div}(\Delta))^2$, so the condition is necessary.

Conversely, assume that $\phi_\Delta$ is injective on $(\text{Div}(\Delta))^2$. First, $\phi_\Delta$ must be injective on $\mathbb{I}_\mathcal{C}$, which is included in $(\text{Div}(\Delta))^2$, and, therefore, on $\text{Obj}(\mathcal{C})$, that is, $\Delta$ must be target-injective. Next, as $\mathbb{I}_\mathcal{C}$ is included in $\text{Div}(\Delta)$, the assumption implies that $\phi_\Delta$ is injective on $\text{Div}(\Delta)$.

We claim that $\phi_\Delta$ induces a permutation of $\mathcal{C}^\times$. Indeed, as $\phi_\Delta$ is a functor, it maps $\mathcal{C}^\times$ into itself. Now, the assumption that $\Delta$ is a Garside map implies that $\text{Div}(\Delta)$ is bounded by $\Delta$, hence, by Lemma V.2.7, $\phi_\Delta$ induces a surjective map of $\text{Div}(\Delta)$ into itself, hence a permutation of $\text{Div}(\Delta)$ as it is injective on $\text{Div}(\Delta)$. Assume $\epsilon \in \mathcal{C}^\times$. Then $\epsilon$ and $\epsilon^{-1}$ belong to $\text{Div}(\Delta)$, so there exist $s,t \in \text{Div}(\Delta)$ satisfying $\phi_\Delta(s) = \epsilon$ and $\phi_\Delta(t) = \epsilon^{-1}$. As $\Delta$ is target-injective, $st$ is defined. Indeed, let $y$ be the target of $s$ and $x'$ be the source of $t$. As $\phi_\Delta$ is a functor, $\phi_\Delta(y)$ is the target of $\phi_\Delta(x')$, that is, of $\epsilon$, whereas $\phi_\Delta(x')$ is the source of $\phi_\Delta(t)$, that is, of $\epsilon^{-1}$. Hence we have $\phi_\Delta(y) = \phi_\Delta(x')$ and, therefore, $y = x'$, that is, $st$ is defined. The argument for $ts$ is symmetric. So $st$ and $ts$ belong to $(\text{Div}(\Delta))^2$. The assumption that $\phi_\Delta$ is injective on $(\text{Div}(\Delta))^2$ implies $st = 1_\mathcal{C}$ and $ts = 1_\mathcal{C}$. Where $x$ (resp. $y$) is the source (resp. target) of $s$. So $s$ belongs to $\mathcal{C}^\times$, and $\phi_\Delta$ induces a permutation of $\mathcal{C}^\times$. Then, Proposition V.1.59 implies that $\mathcal{C}$ is right-cancellative.

Note that the assumptions of Proposition V.2.34 can be slightly weakened: the only assumptions used in the proof is that $\phi_\Delta$ is injective on $\text{Div}(\Delta)$ and that, for $g$ in $(\text{Div}(\Delta))^2$, the relation $\phi_\Delta(g) \in \mathbb{I}_\mathcal{C}$ implies $g \in \mathbb{I}_\mathcal{C}$. We do not know whether the latter condition can be skipped. \qed

Lemma V.2.38.--- (i) A left-cancellative category that is left-Noetherian and admits left-gcds admits conditional right-lcms.

(ii) In a cancellative category that admits conditional right-lcms, any two elements of $\mathcal{C}$ that admit a common left-multiple admit a right-gcd.

Proof. (i) We first show that every nonempty family of elements of $\mathcal{C}$ sharing the same source admits a left-gcd. Let $S$ be a nonempty family of elements of $\mathcal{C}$ that share the same source. An obvious induction shows that, in $\mathcal{C}$, every finite nonempty family of elements of $\mathcal{C}$ sharing the same source has a left-gcd. For any finite nonempty subset of $S$, choose a left-gcd $g_Y$ for $Y$, and let $S'$ be the family of all elements $g_Y$. As $\mathcal{C}$ is left-Noetherian, there exists an element $g_X$ of $S'$ that is $\prec$-minimal in $S'$. We claim that $g_X$ is a left-gcd for $S$. Indeed, let $g$ be an arbitrary element of $S$. The point is to prove $g_X \leq g$. Now, by construction, we have $g_{X,\cup\{g\}} \leq g_X$ and $g_{X,\cup\{g\}} \leq g_X$. As $g_X$ is $\prec$-minimal in $S'$, we must have $g_{X,\cup\{g\}} = g_X$, whence $g_X \leq g_{X,\cup\{g\}}$, as expected. So $S$ has a left-gcd. Then Lemma II.2.21 guarantees that $\mathcal{C}$ admits conditional right-lcms.

(ii) (See Figure 8.) Let $f,g$ be two elements of $\mathcal{C}$ that admit a common left-multiple, say $f'g = g'f$, hence share the same target. The elements $f'$ and $g'$ admit a common right-multiple, namely $f'g'$, hence they admit a right-lcm, say $f'g'' = g'f''$. By definition of a right-lcm, there exists $h$ satisfying $f = f''h$ and $g = g''h$. By construction, $h$ is a common right-divisor of $f$ and $g$.

Let $\hat{h}$ be a common right-divisor of $f$ and $g$. So there exist $\hat{f},\hat{g}$ satisfying $f = \hat{f}\hat{h}$ and $g = \hat{g}\hat{h}$. Then we have $f'\hat{g}\hat{h} = f'g = g'f = g'\hat{f}\hat{h}$, whence $f'\hat{g} = g'f$ by right-cancelling $\hat{h}$. So $f'\hat{g}$ is a common right-multiple of $f'$ and $g'$, hence it is a
right-multiple of their right-lcm, that is, there exists \( h' \) satisfying \( f' \hat{g} = (f'g'')h' \), whence \( \hat{g} = g''h' \) by left-cancelling \( f' \). We deduce \( g''h = g = \hat{g}h = g''h'\hat{h} \), whence \( h = h'\hat{h} \) by right-cancelling \( \hat{h} \). So \( h \) right-divides \( \hat{h} \), which shows that \( h \) is a right-gcd of \( f \) and \( g \).

\[
\text{Figure 8. Construction of a right-gcd using a right-lcm.}
\]

**Lemma V.3.40.**— If \( \Delta \) is a Garside map in a cancellative category \( \mathcal{C} \), then, for all \( s, t \) in \( \text{Div}(\Delta) \), we have \( s \preceq t \iff \partial s \succeq \partial t \) and \( s \succeq t \iff \tilde{\partial}s \preceq \tilde{\partial}t \).

**Proof.** As \( \partial \) and \( \tilde{\partial} \) are bijective, everything is clear. For instance, assume \( s \preceq t \). Then we have \( st' = t \) for some \( t' \), which necessarily belongs to \( \text{Div}(\Delta) \). Calling \( x \) the common source of \( s \) and \( t \), we deduce

\[
\Delta(x) = s \partial s = t \partial t = st' \partial t,
\]

whence \( \partial s = t' \partial t \), and \( \partial s \preceq \partial t \). The other implications are similar. \( \square \)

**Lemma V.3.46.**— If \( \Delta \) is a Garside map in a cancellative category \( \mathcal{C} \) and \( f, g \) are elements of \( \mathcal{C}(x, -) \), then, for every \( m \) satisfying \( m \geq \max(\sup\Delta(f), \sup\Delta(g)) \) and for every \( h \) in \( \text{Div}(\Delta[m]) \), the following conditions are equivalent:

(i) The element \( h \) is a right-lcm of \( f \) and \( g \);

(ii) The element \( \partial[m]h \) is a right-gcd of \( \partial[m]f \) and \( \partial[m]g \).

**Proof.** Assume first that \( m = 1 \). As \( \text{Div}(\Delta) \) is closed under right-comultiple, \( h \) is a right-lcm of \( f \) and \( g \) if and only if the three relations

\[
f \preceq h, \quad g \preceq h,
\]

and \( \forall h' \in \text{Div}(\Delta) \) \( (f \preceq h' \text{ and } g \preceq h') \) implies \( h \preceq h' \)

are satisfied. By the duality formulas of Lemma V.3.40, this is equivalent to

\[
\partial f \succeq \partial h, \quad \partial g \succeq \partial h, \text{ and } \forall h' \in \text{Div}(\Delta) \left( (\partial f \succeq h' \text{ and } \partial g \succeq h') \text{ implies } \partial h \succeq h' \right),
\]

hence, as \( \text{Div}(\Delta) \) coincides with \( \text{Div}(\Delta) \), to \( \partial h \) being a right-gcd of \( \partial f \) and \( \partial g \).

Assume now that \( m \) is arbitrary. By Proposition V.2.32(iv), \( \Delta[m] \) is a Garside map in \( \mathcal{C} \), and \( \partial[m] \) is the associated duality map. Applying the above result to \( \Delta[m] \) and \( \partial[m] \) gives the expected equivalence.

(Observe that, in the situation of Lemma V.3.46, \( \Delta[m](x) \) is a common right-multiple of \( f \) and \( g \), so every possible right-lcm of \( f \) and \( g \) must lie in \( \text{Div}(\Delta[m](x)) \).) \( \square \)
Solution to selected exercises

Exercice 59 (preserving $\text{Div}(\Delta)$).— Assume that $\mathcal{C}$ is a category, $\Delta$ is a map from $\text{Obj}(\mathcal{C})$ to $\mathcal{C}$ and $\phi$ is a functor from $\mathcal{C}$ into itself that commutes with $\Delta$. Show that $\phi$ maps $\text{Div}(\Delta)$ and $\overline{\text{Div}}(\Delta)$ to themselves.

Solution. Assume $s \in \text{Div}(\Delta)$, say $s \lesssim \Delta(x)$. By (the easy direction of) Lemma II.2.8, this implies $\phi(s) \lesssim \phi(\Delta(x))$. By assumption, the latter is $\Delta(\phi(x))$, so $\phi(s)$ belongs to $\text{Div}(\Delta)$. The argument is similar for $\overline{\text{Div}}(\Delta)$.

Exercice 60 (preserving normality I).— Assume that $\mathcal{C}$ is a cancellative category, $\mathcal{S}$ is a Garside family of $\mathcal{C}$, and $\phi$ is a functor from $\mathcal{C}$ to itself. (i) Show that, if $\phi$ induces a permutation of $\mathcal{S}^2$, then $\phi$ preserves $\mathcal{S}$-normality. (ii) Show that $\phi$ preserves non-invertibility, that is, $\phi(g)$ is invertible if and only if $g$ is.

Solution. (i) Assume that $s_1|s_2$ is $\mathcal{S}$-normal. First, by assumption, $\phi$ maps $\mathcal{S}^2$ to itself, hence $\phi(s_1)$ and $\phi(s_2)$ lie in $\mathcal{S}^2$. Assume that $s$ is an element of $\mathcal{S}$ that satisfies $s \lesssim \phi(s_1)\phi(s_2)$. By Lemma II.2.8, we deduce $\phi^{-1}(s) \lesssim s_1s_2$, whence $\phi^{-1}(s) \lesssim s_1$ as $\phi^{-1}$ maps $\mathcal{S}^2$ into itself and $s_1|s_2$, which is $\mathcal{S}$-normal by assumption, is also $\mathcal{S}^2$-normal by Lemma III.1.10. Reapplying $\phi$, we deduce $s \lesssim \phi(s_1)$, and we conclude that $\phi(s_1)|\phi(s_2)$ is $\mathcal{S}$-normal.

(ii) First, $\phi(g)$ is invertible whenever $g$ is invertible since $\phi$ is a functor. Conversely, assume that $\phi(g)$ is invertible. Let $g_1|\cdots|g_p$ be an $\mathcal{S}^2$-normal decomposition of $g$. As $\phi$ is a functor and preserves normality, $\phi(g_1)|\cdots|\phi(g_p)$ is an $\mathcal{S}^2$-normal decomposition of $\phi(g)$. The assumption that $\phi(g)$ is invertible implies that each of $\phi(g_1), \ldots, \phi(g_p)$ is invertible, and then the assumption that $\phi$ is injective on $\mathcal{S}^2$ implies that $g_1, \ldots, g_p$ are invertible. Hence $g$ is invertible.

Exercice 61 (preserving normality II).— Assume that $\mathcal{C}$ is a left-cancellative category and $\mathcal{S}$ is a Garside family of $\mathcal{C}$ that is right-bounded by a map $\Delta$. (i) Show that $\Delta_\mathcal{S}$ preserves normality if and only if there exists an $\mathcal{S}$-head map $H$ satisfying $H(\phi_\Delta(g)) =^* \phi_\Delta(H(g))$ for every $g$ in $(\mathcal{S}^2)^2$, if and only if, for each $\mathcal{S}$-head map $H$, the above relation is satisfied. (ii) Show that a sufficient condition for $\Delta_\mathcal{S}$ to preserve normality is that $\Delta_\mathcal{S}$ preserves left-gcds on $\mathcal{S}^2$, that is, if $r, s, t$ belong to $\mathcal{S}^2$ and $r$ is a left-gcd of $s$ and $t$, then $\Delta_\mathcal{S}(r)$ is a left-gcd of $\Delta_\mathcal{S}(s)$ and $\Delta_\mathcal{S}(t)$.

Solution. (i) Assume that $\phi_\Delta$ preserves normality, $H$ is a $\mathcal{S}$-head map, and $g_1|g_2$ lies in $(\mathcal{S}^2)^2$. Put $g'_1 = H(g_1g_2)$, and let $g'_2$ satisfy $g'_1g'_2 = g_1g_2$. Then $g'_1|g'_2$ is $\mathcal{S}$-normal, hence, as $\phi_\Delta$ preserves normality, so is $\phi_\Delta(g'_1)|\phi_\Delta(g'_2)$. Moreover, we have $\phi_\Delta(g'_1)\phi_\Delta(g'_2) = \phi_\Delta(g_1g_2)$, hence $\phi_\Delta(g'_1)|\phi_\Delta(g'_2)$ is an $\mathcal{S}$-normal decomposition of $\phi_\Delta(g_1g_2)$. By uniqueness of the head, we must have $H(\phi_\Delta(g_1g_2)) =^* \phi_\Delta(g'_1)$. Conversely, assume that $H$ is a $\mathcal{S}$-head map and $H(\phi_\Delta(g)) =^* \phi_\Delta(H(g))$ holds for every $g$ in $(\mathcal{S}^2)^2$. Let $g_1|g_2$ be an $\mathcal{S}$-normal path. By construction, we have $g_1 =^* H(g_1g_2)$. As $\phi_\Delta$ is a functor, this implies $\phi_\Delta(g_1) =^* \phi_\Delta(H(g_1g_2))$ and $\phi_\Delta(H(g_1g_2)) =^* \phi_\Delta(g_1)$, hence $\phi_\Delta(g_1) = ^* \phi_\Delta(H(g_1g_2))$. As we have $\phi_\Delta(H(g_1g_2)) = ^* H(\phi_\Delta(g_1g_2))$ by assumption, we deduce $\phi_\Delta(g_1) = ^* H(\phi_\Delta(g_1g_2))$. Hence $\phi_\Delta(g_1)$ is an $\mathcal{S}$-head for $\phi_\Delta(g_1g_2)$. As we have $\phi_\Delta(g_1g_2) = \phi_\Delta(g_1)\phi_\Delta(g_2)$, we deduce that $\phi_\Delta(g_1)|\phi_\Delta(g_2)$ is an $\mathcal{S}$-normal decomposition of $\phi_\Delta(g_1g_2)$. Hence $\phi_\Delta$ preserves normality.

(ii) Assume that $\Delta_\mathcal{S}$ preserves left-gcds on $\mathcal{S}^2$. Let $g_1|g_2$ be an $\mathcal{S}$-normal path. Let $x$ be the source of $g_2$. By Proposition V.1.53, $\partial_\mathcal{S}(g_1)$ and $g_2$ are left-coprime, that
Exercice 64 (normal decomposition).— Assume that \( \mathcal{C} \) is a left-cancellative category, \( \Delta \) is a right-Garside map in \( \mathcal{C} \) such that \( \phi_{\Delta} \) preserves normality, and \( f, g \) are elements of \( \mathcal{C} \) such that \( fg \) is defined and \( f \neq \Delta(\gamma) \) holds, say \( ff' = \Delta^m(g) \) with \( m \geq 1 \). Show that \( f' \) and \( g \) admit a left-gcd and that, if \( h \) is such a left-gcd, then concatenating a \( \text{Div}(\Delta) \)-normal decomposition of \( fh \) and a \( \text{Div}(\Delta) \)-normal decomposition of \( h^{-1}g \) yields a \( \text{Div}(\Delta) \)-normal decomposition of \( fg \). [Hint: First show that concatenating \( fh \) and a \( \text{Div}(\Delta^m) \)-normal decomposition of \( h^{-1}g \) yields a \( \text{Div}(\Delta^m) \)-normal decomposition of \( fg \) and apply Exercise 42 in Chapter IV.]

Solution. By Proposition V.1.58, \( \Delta^m \) is a right-Garside map in \( \mathcal{C} \) and, by definition, \( f' \) right-divides an element \( \Delta^m(-) \) so, by Lemma V.2.36, \( f' \) and \( g \) admit a left-gcd. Assume that \( h \) is a left-gcd of \( f' \) and \( g \). Since \( \Delta^m \) is a right-Garside map, \( gh \) and \( \Delta^m(-) \), that is, \( ff' \), admit a left-gcd, which (by Exercise 8(i) in Chapter II) is of the form \( fh' \) with \( h' \) a left-gcd of \( f' \) and \( g \). By uniqueness, we have \( h \equiv h' \), whence \( fh \equiv fh' \). So \( fh \) is a left-gcd of \( fg \) and \( \Delta^m(-) \), hence it is a \( \text{Div}(\Delta^m) \)-head of \( fg \). Hence concatenating \( fh \) and a \( \text{Div}(\Delta^m) \)-normal decomposition of \( h^{-1}g \) yields a \( \text{Div}(\Delta^m) \)-normal decomposition of \( fg \). Then Exercise 42 gives the result.

Exercice 65 (iterated duality).— Assume that \( \Delta \) is a Garside map in a cancellative category \( \mathcal{C} \). (i) Show that \( \hat{\partial}^m(g) = \hat{\partial}^m(g) \Delta^{-1}(\phi_{\Delta}(x)) \) holds for \( m' = m \) and \( g \) in \( \text{Div}(\Delta^m(x)) \). (ii) Show that \( \hat{\partial}^m(g) = \hat{\Delta}_{m' - m}(\phi_{\Delta}^m(y)) \hat{\partial}^m(g) \) holds for \( m' = m \) and \( g \) in \( \text{Div}(\Delta^m(y)) \).

Solution. (i) By definition, we have
\[
g \hat{\partial}^m(g) \Delta^{-1}(\phi_{\Delta}(x)) = \Delta^m(x) \Delta^{-1}(\phi_{\Delta}(x)) = \Delta^m(x) = g \hat{\partial}^m(g),
\]
whence the result by left-cancelling \( g \). The computation is symmetric for (ii).

Chapter VI: Germs

Skipped proofs

Solutions to selected exercises

Exercice 65 (not embedding).— Let \( S \) consist of fourteen elements \( 1, a, \ldots, n \), all with the same source and target, and \( \bullet \) be defined by \( l \bullet x = x \bullet l = x \) for each \( x \), plus \( a \bullet a = f, f \bullet c = g, d \bullet e = h, g \bullet h = i, c \bullet d = j, b \bullet j = k, k \bullet e = m \), and \( a \bullet m = n \). (i) Shows that \( S \) is a germ. (ii) Show that, in \( S \), we have \((a \bullet b) \bullet c \bullet d \bullet e) = i \neq n = a \bullet ((b \bullet (c \bullet d)) \bullet e)\), whereas, in \( \text{Mon}(\mathcal{G}) \), we have \( r1 = m \). Conclude.
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Solution. (i) The only nontrivial triples eligible for (VI.1.6) are \((a, b, j), (f, c, d)\), and \((c, d, e)\), for which (VI.1.6) is actually true. (ii) The equalities in \(\mathcal{S}\) follow from the multiplication table of \(\mathcal{S}\). On the other hand, in \(\text{Mon}(\mathcal{S})\), we find \(\iota t = ((\iota a b) c d) (\iota d e) = \iota a ((\iota b (c d)) d e) = \iota a\) applying associativity. Thus \(\iota\) is not injective, and \(\mathcal{S}^2\) does not embed in \(\text{Mon}(\mathcal{S})\).

Exercise 66 (multiplying by invertible elements).— (i) Show that, if \(\mathcal{S}\) is a left-associative germ, then \(\mathcal{S}\) is closed under left-multiplication by invertible elements in \(\text{Cat}(\mathcal{S})\). (ii) Show that, if \(\mathcal{S}\) is an associative germ, \(s \cdot t\) is defined, and \(t' = \tilde{s}\) holds, then \(s \cdot t'\) is defined as well.

Solution. (i) Assume that \(\epsilon\) admits a left-inverse \(\epsilon'\)—as nothing a priori forces the category \(\text{Cat}(\mathcal{S})\) to be left-cancellative, we have to distinguish between left- and right-inverse—and that \(eg\) is defined for some \(g\) lying in \(\mathcal{S}\). Then we have \(g = \epsilon'eg\), so \(eg\) is a right-divisor of an element of \(\mathcal{S}\), hence is an element of \(\mathcal{S}\) as the latter is closed under right-divisor. Applying this to the case when \(g\) is an identity-element shows that the family of all left-invertible elements is included in \(\mathcal{S}\).

Exercise 67 (atoms).— (i) Show that, if \(\mathcal{S}\) is a left-associative germ, the atoms of \(\text{Cat}(\mathcal{S})\) are the elements of the form \(te\) with \(t\) an atom of \(\mathcal{S}\) and \(e\) an invertible element of \(\mathcal{S}\).

(ii) Let \(\mathcal{T}\) be the germ whose table is shown on the right. Show that the monoid \(\text{Mon}(\mathcal{T})\) admits the presentation \(\langle a, e \mid ea = a, e^2 = 1 \rangle\) (see Exercise 48) and that \(a\) is the only atom of \(\mathcal{T}\), whereas the atoms of \(\text{Mon}(\mathcal{T})\) are \(a\) and \(ae\). (iii) Show that \(\mathcal{S}\) is a Garside germ.

Solution. (i) Assume \(s = te\) with \(t\) an atom of \(\mathcal{S}\) and \(e\) an invertible element in \(\mathcal{S}\). Let \(s_1 \cdots s_p\) be a decomposition of \(s\) in \(\text{Cat}(\mathcal{S})\). As \(\mathcal{S}\) generates \(\text{Cat}(\mathcal{S})\), each element \(s_i\) can be expressed as a product of elements of \(\mathcal{S}\), leading to a new decomposition \(t_1 \cdots t_q\) of \(t\) in terms of elements of \(\mathcal{S}\), whence \(t = t_1 \cdots t_q \epsilon^{-1}\) in \(\text{Cat}(\mathcal{S})\) and, as \(\mathcal{S}\) is left-associative, \(t = \Pi(t_1 \cdots t_q \epsilon^{-1})\) in \(\mathcal{S}\), where \(\Pi\) is the partial map of (VI.1.15). As \(t\) is an atom of \(\mathcal{S}\), at most one of the entries \(t_j\) is non-invertible in \(\mathcal{S}\). Hence at most one of the entries \(s_i\) is non-invertible in \(\text{Cat}(\mathcal{S})\), and \(s\) is an atom in \(\text{Cat}(\mathcal{S})\).

Conversely, assume that \(s\) is an atom in \(\text{Cat}(\mathcal{S})\). As \(\mathcal{S}\) generates \(\text{Cat}(\mathcal{S})\), there exists a decomposition \(s_1 \cdots s_p\) of \(s\) into elements of \(\mathcal{S}\). As \(s\) is an atom in \(\text{Cat}(\mathcal{S})\), at most one entry is not invertible. If every entry is invertible, then \(s\) is invertible, contradicting the assumption. So exactly one entry, say \(s_i\), is not invertible. Let \(t = s_1 \cdots s_{i-1} t s_i\) and \(\epsilon = s_{i+1} \cdots s_p\). As \(s_1, \ldots, s_{i-1}\) are invertible, \(t\) is a right-divisor of \(s\), hence it belongs to \(\mathcal{S}\). Moreover, \(t\) is an atom of \(\mathcal{S}\), since a decomposition of \(t\) with more than one non-invertible entry in \(\mathcal{S}\) would provide a similar decomposition in \(\text{Cat}(\mathcal{S})\), contradicting the assumption. On the other hand, \(\epsilon\) is invertible. So \(s\), which is \(te\), has the expected form.

Exercise 68 (families \(\mathcal{J}\) and \(\mathcal{J}^2\)).— Assume that \(\mathcal{S}\) is a left-associative germ. (i) Show that a path \(s_1|s_2\) of \(\mathcal{S}^2\) is \(\mathcal{S}\)-normal if and only if all elements of \(\mathcal{J}(s_1, s_2)\) are invertible. (ii) Assuming in addition that \(\mathcal{S}\) is left-cancellative, show that, for \(s_1|s_2\) in \(\mathcal{S}^2\), the family \(\mathcal{J}(s_1, s_2)\) admits common right-multiples if and only if \(\mathcal{J}(s_1, s_2)\) does.
Solution. (ii) Assume that $J_S(g_1, g_2)$ admits common right-multiples, and let $h, h'$ belong to $J_S(g_1, g_2)$. Then $g_1 \cdot h$ and $g_2 \cdot h'$ are defined and belong to $J_S(g_1, g_2)$, hence, by assumption, $g_1 \cdot h$ and $g_1 \cdot h'$ admit a common right-multiple, say $g''$. Then we have $g_1 \preceq_S g_1 \cdot h \preceq_S g$, whence $g_1 \preceq_S g''$. So there exists $h''$ in $S$ satisfying $g'' = g_1 \cdot h''$. By Lemma VI.1.19, $g_1 \cdot h \preceq_S g_1 \cdot h''$ implies $h \preceq_S h''$, and, similarly, we find $h' \preceq_S h''$. So $h''$ is a common right-multiple of $h$ and $h'$ in $S$. Moreover the assumption that $g_1 \cdot h''$ belongs to $J_S(g_1, g_2)$ implies that $h''$ belongs to $J_S(g_1, g_2)$. So $J_S(g_1, g_2)$ admits common right-multiples. The converse implication is similar, actually simpler as no cancellation is needed.

Exercice 69 (positive generators).— Assume that $\Sigma$ is a family of positive generators in a group $\mathcal{G}$ and $\Sigma$ is closed under inverse, that is, $g \in \Sigma$ implies $g^{-1} \in \Sigma$. (i) Show that $\|g\|_\Sigma = \|g^{-1}\|_\Sigma$ holds for every $g$ in $\mathcal{G}$. (ii) Show that $f^{-1} \preceq_\Sigma g^{-1}$ is equivalent to $f \preceq_\Sigma g$.

Solution. (i) An $S$-word $w$ is a minimal length expression for an element $g$ if and only if $w^{-1}$, which is also an $S$-word, is a minimal length expression of $g^{-1}$. (ii) By definition, $f^{-1} \preceq_\Sigma g^{-1}$ is equivalent to $\|f^{-1}\|_\Sigma + \|g^{-1}\|_\Sigma = \|g^{-1}\|_\Sigma$; hence, by (i), to $\|f\|_\Sigma + \|gf^{-1}\|_\Sigma = \|g\|_\Sigma$ and to $\|f\|_\Sigma + \|g\|_\Sigma = \|g\|_\Sigma$. The latter is $f \preceq_\Sigma g$.

Exercice 70 (minimal upper bound).— For $\preceq$ a partial ordering on a family $S'$ and $f, g, h$ in $S'$, say that $h$ is a minimal upper bound, or mub, for $f$ and $g$, if $f \preceq h$ and $g \preceq h$ holds, but there exists no $h'$ with $h' < h$ satisfying $f \preceq h'$ and $g \preceq h'$. Assume that $\mathcal{G}$ is a groupoid, $\Sigma$ positively generates $\mathcal{G}$, and $\mathcal{H}$ is a subfamily of $\mathcal{G}$ that is closed under $\Sigma$-suffix. Show that $\mathcal{H}^E$ is a Garside germ if and only if, for all $f, g, g', g''$ in $\mathcal{H}$ such that $f \circ g$ and $f \circ g'$ are defined and $g''$ is a $\preceq_\Sigma$-mub of $g$ and $g'$, the product $f \circ g''$ is defined.

Solution. By Lemmas VI.2.60 and VI.2.62, the germ $\mathcal{H}^E$ is left-associative, cancellative, and Noetherian. Hence, by Proposition VI.2.44, $\mathcal{H}^E$ is a Garside germ if and only if it satisfies (VI.2.43). Now, by Lemma VI.2.62, for $g, h$ in $\mathcal{H}$, the relation $g \preceq_{\mathcal{H}^E} h$ is equivalent to $g \preceq_\Sigma h$ and, therefore, $g''$ is an mcm of $g$ and $g'$ in $\mathcal{H}^E$ if and only it it is a $\preceq_\Sigma$-mub of $g$ and $g'$. So the condition is a direct reformulation of (VI.2.43).

Chapter VII: Subcategories

Skipped proofs

Lemma VII.1.3.— If $\mathcal{C}_1$ is a subcategory of a left-cancellative category $\mathcal{C}$, we have

(VII.1.4) \[ \mathcal{C}_1 \subseteq \mathcal{C}^e \cap \mathcal{C}_1, \]

with equality if and only if $\mathcal{C}_1$ is closed under inverse in $\mathcal{C}$. For every subfamily $S$ of $\mathcal{C}$, putting $S_1 = S \cap \mathcal{C}_1$ and $S_1^e = S_1 \mathcal{C}_1^e \cup \mathcal{C}_1$, we have

(VII.1.5) \[ S_1^e \subseteq S^e \cap \mathcal{C}_1. \]

If $\mathcal{C}$ has no nontrivial invertible element, then so does $\mathcal{C}_1$, and (VII.1.4)–(VII.1.5) are equalities.
Addenda to “Foundations of Garside Theory”, by Dehornoy et al.

Lemma VII.1.16.— Every subcategory that is closed under left- or under right-divisor in a left-cancellative category $\mathcal{C}$ is closed under $=^\times$.

Proof. Assume that $C_1$ is a subcategory of $\mathcal{C}$, and we have $g \in C_1$ and $g' =^\times g$, say $g' = ge$ with $e$ invertible. If $C_1$ is closed under left-divisor, we can write $g = g' e^{-1}$, and $g'$ is a left-divisor of $g$, hence it belongs to $C_1$.

On the other hand, assume that $C_1$ is closed under right-divisor. Let $y$ be the target of $g$, and $g'$ be that of $g'$. The assumption that $g$ lies in $C_1$ implies that $y$ lies in $\text{Obj}(C_1)$, hence $1_y$ lies in $C_1$. Now $e^{-1}$ is a right-divisor of $g$, hence it lies in $C_1$, its source $g'$ lies in $\text{Obj}(C_1)$ and $1_y$ lies in $C_1$. Now, we have $1_y = e^{-1} e$, so $e$, a right-divisor of $1_y$, must lie in $C_1$. Hence $g'$, that is, $ge$, lies in $C_1$. □

Lemma VII.1.17.— If $\mathcal{C}, \mathcal{C}'$ are left-cancellative categories, $\phi : \mathcal{C} \to \mathcal{C}'$ is a functor, and $C'_1$ is a subcategory of $\mathcal{C}'$ that is closed under left-divisor (resp. right-divisor), then so is $\phi^{-1}(C'_1)$.

Proof. Assume that $C'_1$ is closed under left-divisor. Put $C_1 = \phi^{-1}(C'_1)$, and assume $f \preceq g \in C_1$. By definition, there exists $g'$ satisfying $fg' = g$, which implies $\phi(f) \phi(g') = \phi(g)$, whence $\phi(f) \preceq \phi(g)$. By assumption, $\phi(g)$ lies in $C'_1$, hence so does $\phi(f)$ as $C'_1$ is closed under left-divisor. Hence $\phi(f)$ belongs to $C'_1$, and $f$ lies in $C_1$. So $C_1$ is closed under left-divisor. The argument when $C'_1$ is closed under right-divisor is symmetric. □

Proposition VII.2.16 (recognizing compatible).— If $S$ is a Garside family in a left-cancellative category $\mathcal{C}$ and $C_1$ is a subcategory of $\mathcal{C}$ that is closed under right-quotient in $\mathcal{C}$, then $C_1$ is compatible with $S$ if and only if

\begin{align*}
\text{(VII.2.17)} & \quad \text{The family } S_1^2 \text{ generates } C_1, \text{ where we put } S_1 = S \cap C_1 \text{ and } S_1^2 = S_1 C_1^e \cup C_1^e, \\
\text{(VII.2.18)} & \quad \text{Every element of } (S_1^2)^2 \text{ admits an } S\text{-normal decomposition with entries in } S_1^2.
\end{align*}

Proof. If $C_1$ is compatible with $S$, then, by Proposition VII.2.14, (VII.2.15) holds. This implies in particular that every element of $C_1$ admits a decomposition with entries in $S_1^2$, hence that $S_1^2$ generates $C_1$, which is (VII.2.17). On the other hand, applying (VII.2.15) to an element of $(S_1^2)^2$ gives (VII.2.18).

Conversely, assume that (VII.2.17) and (VII.2.18) are satisfied. As $C_1$ is closed under right-quotient, it is closed under inverse, $S_1^2 \cap C_1^e = C_1^e$ holds, hence so does
$S_1^i(S_i^1 \cap C^o) \subseteq S_1^i$. Then Lemma VII.2.19 is valid for $S_1^i$, so every element of the subcategory of $C$ generated by $S_1^i$, hence of $C_1$, admits an $S$-normal decomposition whose entries lie in $S_1^i$. In this case, (VII.2.15) is satisfied, and, by Proposition VII.2.14, $C_1$ is compatible with $S$.

**Lemma VII.2.19.**— Assume that $S$ is a Garside family in a left-cancellative category $C$ and $S'$ is a subfamily of $S^2$ such that $S'(S' \cap C^o) \subseteq S'$ holds and every element of $S'^2$ admits a $S$-normal decomposition with entries in $S'$. Then every element in the subcategory of $C$ generated by $S'$ admits a $S$-normal decomposition with entries in $S'$.

**Proof.** First we claim that every element $g$ of $S'^2$ admits a length two $S$-normal decomposition with entries in $S'$: indeed, as $S'$ is included in $S^2$, the $S$-length of $g$ is at most two, so the possible entries at position 3 and beyond in an $S$-normal decomposition of $g$ must be invertible, and the assumption $S'(S' \cap C^o) \subseteq S'$ implies that the latter can be incorporated in the second entry, yielding an $S$-normal decomposition of length two.

Then the argument is exactly the same as for Proposition III.1.49 (left multiplication), except that, here, we use two different reference families, namely $S'$ for the entries of the decomposition and $S$ for the greedy property. The point is to prove that, for every $p$, every element of $S'^p$ admits an $S$-normal decomposition of length $p$ with entries in $S'$. We use induction on $p \geq 1$. For $p = 1$, the result is obvious as $S'$ is included in $S^2$, and, for $p = 2$, the result is the assumption. Let $g$ belong to $S'^p$ with $p \geq 3$. Write $g = sg'$ with $s \in S'$ and $g' \in S'^{p-1}$. By induction hypothesis, $g'$ admits an $S$-normal decomposition $s'_1 \cdots |s'_{p-1}$ with entries in $S'$. Starting with $s_0 = s$, and applying the assumption $p - 1$ times, we find an $S$-normal decomposition $g, s_1 \cdots |s_{p-1} s_p$ is an $S$-normal decomposition of $g$ with entries in $S'$.

**Lemma VII.4.7.**— If $S$ is any subfamily of a left-cancellative category $C$, the identity-functor on $C$ is correct for inverses (resp. right-comultiples, resp. right-complements, resp. right-diamonds) on $S$ if and only if $S$ is closed under inverse (resp. right-comultiple, resp. right-complement, resp. right-diamond) in $C$.

**Proof.** The identity-functor on $C$ is correct for inverses on $S_1$ if and only if, for every $s$ in $S_1$ that is invertible, $s^{-1}$ belongs to $S_1$. By definition, this means that $S_1$ is closed under inverse in $C$.

Next, the identity-functor on $C$ is correct for right-comultiples on $S_1$ if and only if, when $s, t$ lie in $S_1$ and $sg = tf$ holds in $C$, there exist $s', t', h$ satisfying $st' = ts'$, $f = sg'$, and $g = th$, plus $st' \in S_1$. By definition, this means that $S_1$ is closed under right-comultiple in $C$. The result is similar with right-complements and right-diamonds.

**Solutions to selected exercises**

**Exercise 72 (\(\simeq^*\)-closed subcategory).**— (i) Show that a subcategory $C_1$ of a left-cancellative category $C$ is \(\simeq^*\)-closed if and only if, for each $x$ in $\text{Obj}(C_1)$, the
families $C^\circ(x, -)$ and $C^\circ(-, x)$ are included in $C_1$. (ii) Deduce that $C_1$ is $=^*$-closed if and only if $C_1^\circ$ is a union of connected components of $C^\circ$.

Solution. Assume that $C_1$ is $=^*$-closed and $x$ lies in $\text{Obj}(C_1)$. Then $1_x$ belongs to $C_1$. If $\epsilon$ belongs to $C^\circ(x, -)$, we have $\epsilon =^* 1_x$, whence $\epsilon \in C_1$. On the other hand, if $\epsilon$ belongs to $C^\circ(y, x)$, then $\epsilon^{-1}$ belongs to $C^\circ(x, y)$. By the above result, $\epsilon^{-1}$ belongs to $C_1$. It follows that $y$, the target of $\epsilon^{-1}$, lies in $\text{Obj}(C_1)$, and, therefore, $\epsilon$ belongs to $C_1$. Conversely, assume that $C^\circ(x, -)$ is included in $C_1$ for every $x$ in $\text{Obj}(C_1)$, and let $g$ belong to $C_1$ and $g' =^* g$ hold, say $g' = ge$ with $\epsilon \in C^\circ$. Let $x$ be the target of $g$. Then $x$ belongs to $\text{Obj}(C_1)$, hence $\epsilon$ belongs to $C_1$, so $g \in C_1$ implies $g'ge \in C_1$.

Exercice 73 (greedy paths).— Assume that $C$ is a cancellative category, $S$ is included in $C$, and $C_1$ is a subcategory of $C$ that is closed under left-quotient. Put $S_1 = S \cap C_1$. Show that every $C_1$-path that is $S$-greedy in $C$ is $S_1$-greedy in $C_1$.

Solution. With the notation of the proof of Lemma VII.2.1, we have also $f' = f''g_2$ with $f'$ and $g_2$ in $C_1$. This implies $f'' \in S_1$ as $C$ is right-cancellative and $C_1$ is closed under left-quotient in $C$.

Exercice 74 (compatibility with $C$).— Assume that $C$ is a left-cancellative category. Show that every subcategory of $C$ that is closed under inverse is compatible with $C$ viewed as a Garside family in itself.

Solution. Let $C_1$ be a subcategory of $C$ that is closed under inverse. Then $C \cap C_1 = C_1$ is a Garside family in $C_1$. A $C_1$-path $g_1|\cdots|g_q$ is $C$-normal if and only if $g_2, \ldots, g_p$ belong to $C^\circ$, hence if and only if $g_2, \ldots, g_p$ belong to $C_1^\circ$, hence if and only if $g_1|\cdots|g_q$ is $C_1$-normal.

Exercice 75 (not compatible).— Let $M$ be the free Abelian monoid generated by $a$ and $b$, and let $N$ be the submonoid generated by $a$ and $ab$. (i) Show that $N$ is not closed under right-quotient in $M$. (ii) Let $S = \{1, a, b, ab\}$. Show that $N$ is not compatible with $S$. (iii) Let $S' = \{a^ib^j \mid i, j \geq 0, i \in \{0, 1\}\}$. Show that $N$ not compatible with $S'$.

Solution. (i) The elements $ab$ and $a$ lie in $N$, but the right-quotient $b$ does not.

(ii) The family $S \cap N$ is not a Garside family in $N$, because $a$ and $ab$, which belong to $S \cap N$, have no common right-multiple belonging to $S \cap N$.

(iii) The family $S' \cap N$ is a Garside family in $M$, but the $S'$-normal decomposition of $a^ib^j$ is $a^ib^j|b$, whereas the $(S' \cap N)$-normal decomposition of $a^ib^j$ in $N$ is $ab|a$.ab.

Exercice 76 (not closed under right-quotient).— (i) Show that every sub-monoid $mN$ of the additive monoid $N$ is closed under right-quotient, but that $2N + 3N$ of $N$ is not. (ii) Let $M$ be the monoid $N \times (\mathbb{Z}/2\mathbb{Z})^2$, where the generator $a$ of $N$ acts on the generators $e, f$ of $(\mathbb{Z}/2\mathbb{Z})^2$ by $ae = fa$ and $af = ea$, and let $N$ be the submonoid of $M$ generated by $a$ and $e$. Show that $M$ is left-cancellative, and its elements admit a unique expression of the form $a^pe^qf^l$ with $p \geq 0$ and $i, j \in \{0, 1\}$, and that $N$ is $M \setminus \{f, ef\}$. (iii) Show that $N$ is not closed under right-quotient in $M$. (iv) Let $S = \{a\}$. Show that $S$ is a Garside family in $M$ and determine $S^\circ$. Show that $N$ is compatible with $S$. [Hint: Show that $S \cap N$, which is $S$, is not a Garside family in $N$.] (v) Shows that $S^\circ \cap N$ is a Garside family in $N$ and $N$ is compatible with $S^\circ$.  

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Solution. (i) If $f$ and $f + g'$ are multiples of $m$, then $g'$ is a multiple of $m$ as well. On the other hand, $2\mathbb{N} + 3\mathbb{N}$ contains 2 and 3, but does not contain 1 although $3 = 2 + 1$ holds.

(ii) The set $M^\times$ consists of 1, e, f, ef, whereas $N^\times$ consists of 1 and e, so $N$ is closed under inverse in $M$. On the other hand, as a and af belong to $N$ but $f$ does not, $N$ is not closed under right-quotient in $M$.

(iv) We have $S^f = \{a^pe^i f^j | p, i, j \leq 1\}$ (eight elements). Then af, which is ea, belongs to $N^\times S$ but not to $SN^\times \cup N^\times$. So $S \cap N$, which is $S$, is not a Garside family in $N$, and the submonoid $N$ is not compatible with $S$.

(v) Consider $S^f$, which is also a Garside family in $M$. Then $S^f \cap N = S^f \setminus \{f, ef\}$ (six elements), and it is equal to $(S^f \cap N)N^\times \cup N^\times$. Hence $S^f \cap N$ generates $N$. Next, $M$ and $N$ are Noetherian, and both admit right-lcms. Now a direct inspection shows that $S^f \cap N$ is (weakly) closed under right-lcm and right-divisor in $N$ (here we consider only right-divisors that belong to $N$, so $f$, which is a right-divisor of af in $M$, is excluded). Hence, by Corollary IV.2.29 (recognizing Garside, right-lcm case), $S^f \cap N$ is a Garside family in $N$. Finally, a pair $a^{p_0} e^{i_1} f^{j_1} | a^{p_2} e^{i_2} f^{j_2}$ is $S$-normal in $M$ if and only if we do not have $p_1 = 0$ and $p_2 = 1$ and the same condition characterizes $(S^f \cap N)$-normal pairs in $N$. Hence $N$ is compatible with $S^f$.

Exercise 77 (not closed under divisor).— Let $M = \langle a, b \mid ab = ba, a^2 = b^2 \rangle$, and let $N$ be the submonoid of $M$ generated by $a^2$ and $ab$. Show that $N$ is compatible with every Garside family $S$ of $M$, but that $M$ is not closed under left- and right-divisor.

Solution. As seen in Example IV.2.34 (no proper Garside), the only Garside family in $M$ is $M$ itself.

Exercise 78 (head implies closed).— Assume that $C$ is a left-cancellative category, $S$ is a subfamily of $C$ that is closed under right-comultiple in $C$, and $C_1$ is a subcategory of $C$. Put $S_1 = S \cap C_1$. (i) Show that, if every element of $S$ admits a $C_1$-head that lies in $S_1$, then $S_1$ is closed under right-comultiple in $C$. (ii) Show that, if, moreover, $S_1$ is closed under right-quotient in $C$, then $S_1$ is closed under right-diamond in $C$.

Solution. (See Figure 9.) (i) Assume $sg = tf$ with $s, t \in S_1$. As $S$ is closed under right-comultiple, there exist $s', t'$ satisfying $s t' = t s' \leq sg$ with $s t' \in S$. Let $r$ be a $C_1$-head of $s t'$ that lies in $S_1$. As $s$ lies in $C_1$ and $s t'$ lies in $S$, the relation $s \leq st'$ implies $s \leq r$, so we have $r = st'$ for some $t'$ in $C_1$. Similarly, we have $r = st_1$ for some $s_1$ in $C_1$, and, therefore, $r$ witnesses that $S_1$ is closed under right-comultiple in $C$. (ii) If, in addition, $S_1$ is closed under right-quotient in $C_1$, then $s_1$ and $t_1$ must lie in $S_1$ since $s, t$, and $r$ do, and $S_1$ is closed under right-diamond in $C$.

Exercise 79 (head on generating family).— Assume that $C$ is a left-cancellative category that is right-Noetherian, $C_1$ is a subcategory of $C$ that is closed under inverse, and $S$ is a subfamily of $C$ such that every element of $S$ admits a $C_1$-head that lies in $S$. Assume moreover that $S$ is closed under right-comultiple and that $S \cap C_1$ generates $C_1$ and is closed under right-quotient in $C$. Show that $C_1$ is a head-subcategory of $C$. [Hint: Apply Exercise 78.]

Solution. Exercise 78 implies that $S \cap C_1$ is closed under right-diamond in $C$. Then Proposition IV.1.15 (factorization grid) implies that the subcategory of $C$
generated by \( S \cap C_1 \), which is \( C_1 \) by assumption, is closed under right-diamond in \( C \). By Lemma VII.1.8, it follows that \( C_1 \) is closed under right-quotient in \( C \). Then, as \( C \) is right-Noetherian, Proposition VII.1.21 implies that \( C_1 \) is a head-subcategory of \( C \).

**Exercise 80 (transitivity of compatibility).**— Assume that \( S \) is a Garside family in a left-cancellative category \( C \), \( C_1 \) is a subcategory of \( C \) that is compatible with \( S \), and \( C_2 \) is a subcategory of \( C_1 \) that is compatible with \( S_1 = S \cap C_1 \). Show that \( C_2 \) is compatible with \( S \).

**Solution.** Put \( S_2 = S_1 \cap C_2 \). By assumption, we have \( C_2^\prime = C_2 \cap C_2 = (C_1 \cap C_2) \cap C_2 = C_1 \cap C_2 \). Therefore, \( C_2 \) is closed under inverse in \( C \). Next, as \( C_1 \) is compatible with \( S \), the family \( S_1 \) is a Garside family in \( C_1 \). Now, as \( C_2 \) is compatible with \( S_1 \) in \( C_1 \), the family \( S_2 \), which is \( S_1 \cap C_2 \), is a Garside family in \( C_2 \). Finally, as \( S_2 = S_1 \cap C_2 \) holds, an \( C_2 \) path is \( S_2 \)-normal in \( C_2 \) if and only if it is \( S_1 \)-normal in \( C_1 \), hence if and only if it is \( S \)-normal in \( C \). Hence \( C_2 \) is compatible with \( S \).

**Exercise 81 (transitivity of head-subcategory).**— Assume that \( C \) is a left-cancellative category, \( C_1 \) is a head-subcategory of \( C \), and \( C_2 \) is a subcategory of \( C_1 \). Show that \( C_2 \) is a head-subcategory of \( C \) if and only if it is a head-subcategory of \( C_1 \).

**Solution.** Assume that \( C_2 \) is a head-subcategory of \( C \). If \( \epsilon \) belongs to \( C_1^\ast \cap C_2 \), then it belongs to \( C_1^\ast \cap C_2 \), hence \( \epsilon^{-1} \) belongs to \( C_2 \), so \( C_2 \) is closed under inverse in \( C_1 \). Moreover, every element of \( C_1 \) admits a \( C_2 \)-head since every element of \( C \) does. Conversely assume that \( C_2 \) is a head-subcategory of \( C_1 \). If \( \epsilon \) belongs to \( C_1^\ast \cap C_2 \), then it belongs to \( C_1^\ast \cap C_1 \), hence to \( C_1^\ast \) since \( C_1 \) is a head-subcategory of \( C \). So \( \epsilon \) belongs to \( C_1^\ast \cap C_2 \), hence to \( C_2 \) since \( C_2 \) is a head-subcategory of \( C_1 \). So \( C_2 \) is closed under inverse in \( C \). Next assume \( g \in C \). Let \( g' \) be a \( C_1 \)-head of \( g \) in \( C \), and \( g'' \) be a \( C_2 \)-head of \( g' \) in \( C_1 \). Assume \( h \in C_2 \) and \( h \leq g \). As \( h \) belongs to \( C_1 \), we must have \( h \leq g' \). Then, as \( h \) belongs to \( C_2 \), we must have \( h \leq g'' \). So \( g'' \) is a \( C_2 \)-head of \( g \), and \( C_2 \) is a head-subcategory of \( C \).

**Exercise 82 (recognizing compatible IV).**— Assume that \( S \) is a Garside family in a left-cancellative category \( C \) and \( C_1 \) is a subcategory of \( C \) that is closed under right-quotient in \( C \). Show that \( C_1 \) is compatible with \( S \) if and only if, putting \( S_1 = S \cap C_1 \), (i) the family \( S_1 \) is a Garside family in \( C_1 \), and (ii) a \( C_1 \)-path is strictly \( S_1 \)-normal in \( C_1 \) if and only if it is strictly \( S \)-normal in \( C \).

**Figure 9. Solution to Exercise 79**
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Solution. Assume that $C_1$ is compatible with $S$. By definition, $S_1$ is a Garside family in $C_1$, so (i) is satisfied. Next, assume that $g_1|\cdots|g_q$ is a strict $S$-normal $C_1$-path. By assumption, $g_1|\cdots|g_q$ is an $S_1$-normal path in $C_1$. Moreover, by definition, $g_1,\ldots,g_q$ are not invertible in $C$, hence they are not invertible either in $C_1$, and $g_1,\ldots,g_{q-1}$ belong to $S$ and to $C_1$, hence to $S_1$. So $g_1|\cdots|g_q$ is strictly $S_1$-normal in $C_1$. Conversely, assume that $g_1|\cdots|g_q$ is strictly $S_1$-normal in $C_1$. As $C_1$ is compatible with $S$, the path $g_1|\cdots|g_q$ is $S$-normal in $C$. Moreover, the assumption that $g_1,\ldots,g_{q-1}$ belong to $S_1$ implies that they belong to $S$. Finally, the assumption that no $g_i$ is invertible in $C_1$ implies that they are not invertible in $C$ either. So (ii) is satisfied.

Conversely, assume that $C_1$ satisfies (i) and (ii). Assume that $g_1|g_2$ is an $S_1$-path. Then $g_1g_2$ is invertible in $C_1$ if and only if it is invertible in $C$. In this case, $g_1|g_2$ is both $S_1$-normal in $C_1$ and $S$-normal in $C$. Otherwise, if $g_1g_2$ belongs to $S_1C_1^\prime$, then $g_1|g_2$ is $S_1$-normal in $C_1$, if and only if $g_2$ is invertible in $C_1$ and, similarly, it is $S$-normal in $C$ if and only if $g_2$ is invertible in $C$, so both conditions are equivalent. Finally, if $g_1|g_2$ has $S_1$-length 2, it admits a strict $S_1$-normal decomposition in $C_1$, say $h_1|h_2$. Then $g_1|g_2$ is $S_1$-normal in $C_1$ if and only if there exist $\epsilon$ invertible in $C_1$ satisfying $g_1 = h_1\epsilon$ and $h_2 = \epsilon g_2$; similarly, $g_1|g_2$ is $S$-normal in $C$ if and only if there exist $\epsilon$ invertible in $C$ satisfying $g_1 = h_1\epsilon$ and $h_2 = \epsilon g_2$; both conditions are equivalent again. So an $S_1$-path is $S_1$-normal in $C_1$ if and only if it is $S$-normal in $C$, and, by definition, $C_1$ is compatible with $S$.

**Exercise 83 (inverse image).** Assume that $C, C'$ are left-cancellative categories, $\phi$ is a functor from $C$ to $C'$, and $C_1'$ is a subcategory of $C'$ that is closed under left- and right-divisor. Show that the subcategory $\phi^{-1}(C_1')$ is compatible with every Garside family of $C$. (ii) Let $B^\prime$ be the Artin–Tits monoid of type $B$ as defined in Example VII.4.21. Show that the map $\phi$ defined by $\phi(\sigma_0) = 1$ and $\phi(\sigma_i) = 0$ for $i \geq 1$ extends into a homomorphism of $B^\prime$ to $\mathbb{N}$, and that the submonoid $N = \{ g \in M \mid \phi(g) = 0 \}$ of $B^\prime$ is compatible with every Garside family of $B^\prime$.

Solution. (i) By Proposition VII.1.18, the subcategory $\phi^{-1}(C_1')$ is closed under right-quotient and under $\Rightarrow$. Then apply Proposition VII.2.21. (ii) Use (i)

**Exercise 84 (intersection).** Assume that $S$ is a Garside family in a left-cancellative category $C$. (i) Let $F$ be the family of all subcategories of $C$ that are closed under right-quotient, compatible with $S$, and $\Rightarrow$-closed. Show that every intersection of elements of $F$ belongs to $F$. (ii) Same question when “$\Rightarrow$-closed” is replaced with “including $C'$”. (iii) Same question when $C$ contains no nontrivial invertible element and “$\Rightarrow$-closed” is skipped.

Solution. (i) Let $(C_i)_{i \in I}$ be a family of $\Rightarrow$-closed subcategories of $C$ that are compatible with $S$. First, an intersection of subcategories is a subcategory. Next, an intersection of $\Rightarrow$-closed families is $\Rightarrow$-closed and, similarly, an intersection of families that are closed under right-quotient is closed under right-quotient. Finally, let $g$ belong to $\bigcap C_i$. Then $g$ admits an $S$-normal decomposition $s_1|\cdots|s_p$. By Lemma VII.2.20, the latter is $(S \cap C_i)$-normal for every $i$, hence it its entries lie in every subfamily $C_i$, hence in their intersection. By Proposition VII.2.21 we deduce that $\bigcap C_i$ is compatible with $S$.

(ii) An intersection of families that include $C'$ includes $C'$. On the other hand, a subcategory that includes $C'$ is $\Rightarrow$-closed and we apply (i).
(iii) In this case, every subcategory is \( =^\circ \)-closed, and we apply (i).

**Exercice 85 (fixed points).**— Assume that \( \mathcal{C} \) is a left-cancellative category and \( \phi : \mathcal{C} \to \mathcal{C} \) is a functor. Show that the fixed point subcategory \( \mathcal{C}^\circ \) is compatible with \( \mathcal{C} \) viewed as a Garside family in itself.

**Solution.** First, assume that \( \epsilon \) belongs to \( \mathcal{C}(x, y) \cap \mathcal{C}^\circ \). Necessarily we have \( \phi(x) = x \). As \( \phi \) is a functor, we find \( \epsilon^{-1} = 1_\epsilon = \phi(1_\epsilon) = \phi(\epsilon \epsilon^{-1}) = \phi(\epsilon) \phi(\epsilon^{-1}) = \epsilon \phi(\epsilon^{-1}) \), whence \( \epsilon^{-1} = \phi(\epsilon^{-1}) \), so \( \mathcal{C}^\circ \cap \mathcal{C}^\circ \subseteq \mathcal{C}(\mathcal{C}^\circ) \cap \mathcal{C}(\mathcal{C}^\circ) \) holds. Next, we have \( \mathcal{C}^\circ = \mathcal{C} \), so \( \mathcal{C}^\circ \cap \mathcal{C}^\circ = \mathcal{S}^\circ \) trivially holds, and \( \mathcal{C}^\circ \) and \( \mathcal{C} \) satisfy (the counterpart of) (VII.2.11). On the other hand, every element \( g \) of \( \mathcal{C} \) admits the \( \mathcal{C} \)-normal decomposition \( g_1 \), so, in particular, every element of \( \mathcal{C}^\circ \) has a \( \mathcal{C} \)-normal decomposition whose entries lie in \( \mathcal{C}^\circ \). So (the counterpart of) (VII.2.12) is satisfied and, by Proposition VII.2.10, \( \mathcal{C}^\circ \) is compatible with \( \mathcal{C} \).

**Exercice 86 (connection between closure properties).**— Assume that \( \mathcal{S} \) is a subfamily in a left-cancellative category \( \mathcal{C} \) and \( \mathcal{S}_1 \) is a subfamily of \( \mathcal{S} \). (i) Show that, if \( \mathcal{S}_1 \) is closed under product, inverse, and right-complement in \( \mathcal{S} \), then \( \mathcal{S}_1 \) is closed under right-quotient in \( \mathcal{S} \). (ii) Assume that \( \mathcal{S}_1 \) is closed under product and right-complement in \( \mathcal{S} \). Show that, if \( \mathcal{S} \) is closed under left-divisor in \( \mathcal{C} \), then \( \mathcal{S}_1 \) is closed under right-multiplie in \( \mathcal{S} \). Show that, if \( \mathcal{S} \) is closed under right-diamond in \( \mathcal{C} \), then \( \text{Sub}(\mathcal{S}_1) \) is closed under right-multiplie in \( \mathcal{S} \). (iii) Show that, if \( \mathcal{S}_1 \) is closed under identity and product in \( \mathcal{S} \), then \( \mathcal{S}_1 \) is closed under inverse and right-diamond in \( \mathcal{S} \) if and only if \( \mathcal{S}_1 \) is closed under right-quotient and right-multiplie in \( \mathcal{S} \).

**Solution.** (i) The argument is the same as for Lemma VII.1.8. Assume \( sg = t \) with \( s, t \) in \( \mathcal{S}_1 \) and \( g \) in \( \mathcal{S} \). As \( \mathcal{S}_1 \) is closed under right-complement in \( \mathcal{S} \), there exist \( s', t' \) in \( \mathcal{S}_1 \) and \( r \) in \( \mathcal{S} \) satisfying \( st' = ts' \), \( 1_y = s'r \), and \( g = t'r \), where \( y \) is the target of \( t \). The second equality implies that \( s' \) is invertible, and the assumption that \( \mathcal{S}_1 \) is closed under inverse in \( \mathcal{S} \) then implies that \( s'^{-1} \), that is, \( r \), lies in \( \mathcal{S}_1 \). Hence \( t'r \), that is \( g \), belongs to \( \mathcal{S}_1^{[\mathcal{S}]} \cap \mathcal{S} \), hence to \( \mathcal{S}_1 \) as the latter is closed under product in \( \mathcal{S} \). So \( \mathcal{S}_1 \) is closed under right-quotient in \( \mathcal{S} \).

(iii) Assume that \( \mathcal{S}_1 \) is closed under inverse and right-diamond in \( \mathcal{S} \). Then, by definition, \( \mathcal{S}_1 \) is closed under right-multiplie and right-complement, and, by Exercice 89 (transfer of closure), it is closed under right-quotient.

Conversely, assume that \( \mathcal{S}_1 \) is closed under right-quotient and right-multiplie in \( \mathcal{S} \). First, the assumption that \( \mathcal{S}_1 \) is closed under right-quotient trivially implies that \( \mathcal{S}_1 \) is closed under inverse. Next, by the same argument as in Lemma IV.1.8, the assumption that \( \mathcal{S}_1 \) is closed under right-multiplie and right-quotient implies that it is closed under right-diamond.

**Exercice 87 (subgerm).**— Assume that \( \mathcal{S} \) is a left-cancellative germ and \( \mathcal{S}_1 \) is a subgerm of \( \mathcal{S} \) such that the relation \( \leq_{\mathcal{S}_1} \) is the restriction to \( \mathcal{S}_1 \) of the relation \( \leq_{\mathcal{S}} \). Show that \( \mathcal{S}_1 \) is closed under right-quotient in \( \mathcal{S} \).

**Solution.** Assume \( f \bullet h = g \) with \( f, g \in \mathcal{S}_1 \). By definition, \( f \leq_{\mathcal{S}} g \) holds, hence so does \( f \leq_{\mathcal{S}_1} g \). This means that there exists \( h_1 \) in \( \mathcal{S}_1 \) satisfying \( f \bullet h_1 = g \). The assumption that \( \mathcal{S} \) is left-cancellative implies \( h_1 = h \), whence \( h \in \mathcal{S} \). So \( \mathcal{S}_1 \) is closed under right-quotient in \( \mathcal{S} \).
Exercice 88 (transitivity of closure).— Assume that $S$ is a subgerm of a Garside germ $S$, the subcategory $\text{Sub}(S_1)$ is closed under right-quotient and right-diamond in $\text{Cat}(S)$, and $S_1$ is closed under inverse and right-complement (resp. right-diamond) in $\text{Sub}(S_1)$. Show that $S_1$ is closed under inverse and right-complement (resp. right-diamond) in $S$.

Solution. By Lemma VII.1.7, the assumption that $\text{Sub}(S_1)$ is closed under right-quotient implies that it is closed under inverse, and Lemma VII.3.7, which is valid since the assumption that $S$ is a Garside germ implies that $S$ is closed under right-quotient in $\text{Cat}(S)$, then implies that $S_1$ is closed under inverse in $S$. Next, by Lemma VII.3.8 applied with $\text{Sub}(S_1)$ in place of $S$, the assumption that $S_1$ is closed under right-complement (resp. right-diamond) in $\text{Sub}(S_1)$ implies that $S_1$ is closed under right-complement (resp. right-diamond) in $\text{Cat}(S)$. Now, as $S$ is a solid Garside family in $\text{Cat}(S)$, it is closed under right-divisor in $\text{Cat}(S)$, hence a fortiori under right-quotient. Applying once more Lemma VII.3.8, we deduce from the fact that $S_1$ is closed under right-complement (resp. right-diamond) in $\text{Cat}(S)$ that $S_1$ is closed under right-complement (resp. right-diamond) in $S$.

Exercice 89 (transfer of closure).— Assume that $C$ is a left-cancellative category, $S$ is a subfamily of $C$, and $S_1$ is a subfamily of $S$ that is closed under identity and product. (i) Show that, if $\text{Sub}(S_1)$ is closed under right-quotient in $C$, then $S_1$ is closed under right-quotient in $S$. (ii) Show that, if, moreover, $S$ is closed under right-quotient in $C$, then $S_1$ is closed under right-quotient in $\text{Sub}(S_1)$.

Solution. (i) Assume that $t$ belongs to $S$ and $s$ and $st$ belong to $S_1$. Then $s$ and $st$ belong to $\text{Sub}(S_1)$, so, as the latter is assumed to be closed under right-quotient, $t$ must belong to $\text{Sub}(S_1)$, hence to $\text{Sub}(S_1) \cap S$. By Proposition VII.3.3, the latter is $S_1$. So $S_1$ is closed under right-quotient in $S$.

(ii) Assume that $S$ is closed under right-quotient in $C$. Assume that $t$ belongs to $\text{Sub}(S_1)$ and $s$ and $st$ belong to $S_1$. Then $s$ and $st$ belong to $S$, so the assumption implies that $t$ belong to $S$, hence to $\text{Sub}(S_1) \cap S$, which is $S_1$ by Proposition VII.3.3. So $S_1$ is closed under right-quotient in $\text{Sub}(S_1)$.

Exercice 90 (braid subgerm).— Let $S$ be the six-element Garside germ associated with the divisors of $\Delta_3$ in the braid monoid $B_3^+$. (i) Describe the subgerm $S_1$ of $S$ generated by $\sigma_1$ and $\sigma_2$. Compare $\text{Mon}(S_1)$ and $\text{Sub}(S_1)$ (describe them explicitly). (ii) Same questions with $\sigma_1$ and $\sigma_2 \sigma_1$. Is $S_1$ closed under right-quotient in $S$ in this case?

Solution. (i) The subgerm of $S$ generated by $\sigma_1$ and $\sigma_2$ is the closure of $\{\sigma_1, \sigma_2\}$ under identity and product in $S$; so it contains 1, and, as $\sigma_1 \cdot \sigma_2$ is defined in $S$, it contains $\sigma_1 \cdot \sigma_2$, that is, $\sigma_1 \sigma_2$. Then it contains $\sigma_1 \sigma_2 \sigma_1$, which is $\Delta_3$, etc. Finally, one finds $S_1 = S$, whence $\text{Mon}(S_1) = \text{Sub}(S_1) = B_3^+$.

(ii) The closure of $\{\sigma_1, \sigma_2 \sigma_1\}$ under identity and product is $\{1, \sigma_1, \sigma_2 \sigma_1, \Delta_3\}$. In the associated germ $S_1$, the only nontrivial product is $\sigma_1 \cdot \sigma_2 \sigma_1 = \Delta_3$, so $\text{Mon}(S_1)$ is $(a, b, \Delta_3 | ab = \Delta_3)$, that is, a free monoid based on $a$ and $b$. On the other hand, in $\text{Sub}(S_1)$, the relation $(\sigma_2 \sigma_1)^3 = \Delta_3^2$ holds, corresponding to $b^3 = (ab)^2$, which fails in $\text{Mon}(S_1)$: so $\text{Mon}(S_1)$ is not isomorphic to $\text{Sub}(S_1)$. Here $\text{Sub}(S_1)$ is not closed under right-quotient in $S$, as $\Delta_3$ and $\sigma_2 \sigma_1$ lie in $\text{Sub}(S_1)$, but we have $\Delta_3 = \sigma_2 \sigma_1 \cdot \sigma_2$ in $S$ and $\sigma_2 \notin S_1$. 

Addenda to “Foundations of Garside Theory”, by Dehornoy et al.
Exercice 91 (\(=^\varepsilon\)-closed).— Show that, if \(\mathcal{S}_1\) is a subgerm of an associative germ \(\mathcal{S}\), then \(\text{Sub}(\mathcal{S}_1) = =^\varepsilon\)-closed in \(\text{Cat}(\mathcal{S})\) if and only if \(\mathcal{S}_1\) is \(=_{\mathcal{S}}\)-closed in \(\mathcal{S}\).

Solution. Assume that \(\mathcal{S}_1\) is an \(=_{\mathcal{S}}\)-closed subgerm of \(\mathcal{S}\). Assume that \(g\) is an element of \(\text{Sub}(\mathcal{S}_1)\) and \(g' =^\varepsilon g\) holds in \(\text{Cat}(\mathcal{S})\). By definition, \(g\) admits an \(S_1\)-decomposition, say \(s_1\cdots|s_p\), and, in \(\text{Cat}(\mathcal{S})\), we then have \(g' = s_1\cdots s_p\epsilon\). As \(\mathcal{S}\) is closed under left-divisor in \(\text{Cat}(\mathcal{S})\) since \(\mathcal{S}\) is right-associative, the assumption that \(s_p\) lies in \(\mathcal{S}\) implies that its left-divisor \(s_p\epsilon\) also lies in \(\mathcal{S}\), that is, \(s_p\epsilon\) is defined in \(\mathcal{S}\). The assumption that \(\mathcal{S}_1\) is \(=_{\mathcal{S}}\)-closed in \(\mathcal{S}\) then implies that \(s_p\epsilon\) belongs to \(\mathcal{S}_1\), which implies that \(g'\), which is \(s_1\cdots s_{p-1}(s_p\epsilon)\), lies in \(\text{Sub}(\mathcal{S}_1)\). Conversely, assume that \(\text{Sub}(\mathcal{S}_1)\) is an \(=^\varepsilon\)-closed subcategory of \(\text{Cat}(\mathcal{S})\). Assume that \(s\) belongs to \(\mathcal{S}_1\) and \(s' =_{\mathcal{S}} s\) holds in \(\mathcal{S}\). This means that there exists \(\epsilon\) in \(\mathcal{S}^\varepsilon\) satisfying \(s' = s\epsilon\). Then, in \(\text{Cat}(\mathcal{S})\), we have \(s' = s\epsilon\), whence \(s' =^\varepsilon s\). The assumption that \(\text{Sub}(\mathcal{S}_1)\) is \(=^\varepsilon\)-closed implies \(s' \in \text{Sub}(\mathcal{S}_1)\), whence \(s' \in \text{Sub}(\mathcal{S}_1)\cap \mathcal{S}\). As \(\mathcal{S}_1\) is a subgerm of \(\mathcal{S}\), the latter family is \(\mathcal{S}_1\).

Exercice 92 (correct vs. mcms).— Assume that \(\mathcal{C}\) and \(\mathcal{C}'\) are left-cancellative categories and \(\mathcal{S}\) is included in \(\mathcal{C}\). Assume moreover that \(\mathcal{C}\) and \(\mathcal{C}'\) admit right-mcms and \(\mathcal{S}\) is closed under right-mcm. Show that a functor \(\phi\) from \(\mathcal{C}\) to \(\mathcal{C}'\) is correct for right-comultiples on \(S\) if and only if, for all \(s, t\) in \(S\), every right-mcm of \(\phi(s)\) and \(\phi(t)\) is \(=^\varepsilon\)-equivalent to the image under \(\phi\) of a right-mcm of \(s\) and \(t\).

Solution. Assume that \(\phi\) is correct for right-comultiples on \(S\), that \(s, t\) lie in \(\mathcal{S}\), and that \(h\) is a right-mcm of \(\phi(s)\) and \(\phi(t)\). By definition, there exists a common right-multiple \(r\) of \(s\) and \(t\) that lies in \(\mathcal{S}\) and satisfies \(\phi(r) \preceq h\). As \(\mathcal{C}\) admits right-mcms, there exists a right-mcm \(r'\) of \(s\) and \(t\) satisfying \(r' \preceq r\) and, as \(\mathcal{S}\) is closed under right-mcm, \(r'\) lies in \(\mathcal{S}\). Then \(\phi(r') \preceq h\) holds, and \(\phi(r')\) is a common right-multiple of \(\phi(s)\) and \(\phi(t)\). As \(r\) is minimal, we deduce \(\phi(r') =^\varepsilon h\). So the condition is necessary. Conversely, assume that \(\phi\) preserves mcms in the sense of the statement. Assume that \(s, t\) lie in \(\mathcal{S}\), and \(h\) is a common right-multiple of \(\phi(s)\) and \(\phi(t)\). As \(\mathcal{C}'\) admits right-mcms, \(h\) is a right-multiple of some right-mcm of \(\phi(s)\) and \(\phi(t)\), hence, by (ii), of some element \(\phi(r)\) where \(r\) is a right-mcm of \(s\) and \(t\). By assumption, \(r\) belongs to \(\mathcal{S}\), and there exist \(s', t'\) in \(\mathcal{C}\) satisfying \(st' = ts' = r\). Hence \(\phi\) is correct for right-comultiples on \(\mathcal{S}\).

Chapter VIII: Conjugacy

Skipped proofs

Solution to selected exercises

Exercice 97 (quasi-distance).— (i) In Context VIII.3.7, show that \(\ell_\Delta(g) = \sup_\Delta(g^\epsilon)\) holds for every \(g\) in \(\mathcal{G}\). (ii) For \(g, g'\) in \(\mathcal{G}\), define \(\text{dist}(g, g')\) to be \(\infty\) if \(g, g'\) do not share the same source, and to be \(\ell_\Delta(g^{-1}g')\) otherwise. Show that \(\text{dist}\) is a quasi-distance on \(\mathcal{G}\) that is compatible with \(=_{\Delta}\).
Solution. (ii) The canonical length is invariant under left- and right-multiplication by $\Delta$ and, therefore, dist takes constant values on $\equiv_\Sigma$-classes.

Chapter IX: Braids

Skipped proofs

Solution to selected exercises

Exercice 102 (smallest Garside, right-angled type).— Assume that $B^+$ is a right-angled Artin–Tits monoid, that is, $B^+$ is associated with a Coxeter system $(W, \Sigma)$ satisfying $m_{s,t} \in \{2, \infty\}$ for all $s, t$ in $\Sigma$. (i) For $I \subseteq \Sigma$, denote by $\Delta_I$ the right-lcm (here the product) of the elements of $I$, when it exists (that is, when the elements pairwise commute). Show that the divisors of $\Delta_I$ are the elements $\Delta_J$ with $J \subseteq I$. (ii) Deduce that the smallest Garside family in $B^+$ is finite and consists of the elements $\Delta_I$ for $I \subseteq \Sigma$.

Solution. (i) Clearly $J \subseteq I$ implies $\Delta_J \leq \Delta_I$ and $\Delta_I \trianglerighteq \Delta_J$, since the elements of $I$ commute. Conversely, the point is to see that, if $u$ is a word in $S$ that involves at least one letter not in $I$, then (the class of) $u$ cannot right-divide $\Delta_I$: this is so since, when left-reversing $\Delta_I$, a negative letter $u$ can only vanish when it is adjacent to the positive letter $s$, and it, if $u$ is a word in $I$ in which some letter is repeated twice, then the second occurrence cannot vanish as we are arguing in the free abelian monoid based on $I$.

(ii) Let $S$ be the family of all elements $\Delta_I$ with $I$ a set of pairwise commuting atoms in $B^+$. As the smallest Garside family of $B^+$ includes $I$ and is closed under right-lcm, it must include $S$. On the other hand, $S$ includes $\Sigma$, it is closed under right-lcm by definition, and it is closed under right-divisor by (i). Corollary IV.2.29 (recognizing Garside, right-lcm case) implies that $S$ is a Garside family in $B^+$.

Exercice 103 (smallest Garside, large type).— Assume that $B^+$ is an Artin–Tits monoid of large type, $B^+$ is associated with a Coxeter system $(W, \Sigma)$ satisfying $m_{s,t} \geq 3$ for all $s, t$ in $\Sigma$. Put

\[
\begin{align*}
\Sigma_1 &= \{ s \in \Sigma \mid \forall r \in \Sigma \ (m_{r,s} = \infty) \}, \\
\Sigma_2 &= \{ (s, t) \in \Sigma^2 \mid m_{s,t} < \infty \text{ and } \forall r \in \Sigma \ (m_{r,s} + m_{r,t} = \infty) \}, \\
\Sigma_3 &= \{ (r, s, t) \in \Sigma^3 \mid m_{r,s} + m_{r,t} + m_{s,t} < \infty \}, \\
and E &= \Sigma_1 \cup \{ \Delta_{s,t} \mid (s, t) \in \Sigma_2 \} \cup \{ r\Delta_{s,t} \mid (r, s, t) \in \Sigma_3 \} \text{ (we write $\Delta_{s,t}$ for the right-lcm of $s$ and $t$ when it exists). (i) Explicitly describe the elements of the closure $S$ of $E$ under right-divisor, and deduce that $S$ is finite. (ii) Show that $S$ is closed under right-lcm and deduce that $S$ is a Garside family in $B^+$. (iii) Show that $S$ is the smallest Garside family in $B^+$. (iv) Show that, if $\Sigma$ has $n$ elements and $m_{s,t} \neq \infty$ holds for all $s, t$, then $E$ has $3\binom{n}{3}$ elements. (v) Show that, if $\Sigma$ has $n$ elements and $m_{s,t} = m$ holds for all $s, t$, then $S$ has $(n + 2m - 5)\binom{n}{3} + n + 1$ elements. Apply to $n = m = 3$.
\end{align*}
\]
Addenda to “Foundations of Garside Theory”, by Dehornoy et al. 35

Solution. (i) For \( s \) in \( \Sigma_1 \), the only right-divisors of \( s \) are 1 and \( s \); for \( (s, t) \) in \( \Sigma_2 \), the right-divisors of \( \Delta_{s,t} \) are the elements \( \{s|t|k\} \) and \( \{t|s|k\} \) with \( k < m_{s,t} \); this is so because right-divisors are detected using left-reversing (Section II.4) and it is clear that, if \( u \) is a word involving a letter different from \( s \) and \( t \) or if \( u \) involves two letters \( s \) or two letters \( t \), or if \( u \) is of the form \( \{s|t|k\} \) and \( \{t|s|k\} \) with \( k > m_{s,t} \), then left-reversing \( \Delta_{s,t}u^{-1} \) cannot result in a positive word. Finally, for \( (r, s, t) \) in \( \Sigma_3 \), the right-divisors of \( r\Delta_{s,t} \) are \( r\Delta_{s,t} \) and the elements \( \{s|t|k\} \) and \( \{t|s|k\} \) with \( k < m_{s,t} \); again, this is so because left-reversing \( \Delta_{s,t}u^{-1} \) can result in a positive word only in the cases described above, and it can result in \( r^{-1} \) concatenated with a positive word only if \( u \) is \( r\Delta_{s,t} \) itself.

(ii) We have to analyze when two elements of \( S \) may admit a common right-multiple. As there are three cases for each factor, nine cases are to be considered. Many of them are trivial. First, all cases involving an atom of \( \Sigma \) are multiple. As there are three cases for each factor, nine cases are to be considered. Indeed, \( \Sigma \) is a word involving a letter different from \( s \) and \( t \) or if \( u \) involves two letters \( s \) or two letters \( t \), or if \( u \) is of the form \( \{s|t|k\} \) and \( \{t|s|k\} \) with \( k > m_{s,t} \), then left-reversing \( \Delta_{s,t}u^{-1} \) cannot result in a positive word. Finally, for \( (r, s, t) \) in \( \Sigma_3 \), the right-divisors of \( r\Delta_{s,t} \) are \( r\Delta_{s,t} \) and the elements \( \{s|t|k\} \) and \( \{t|s|k\} \) with \( k < m_{s,t} \); again, this is so because left-reversing \( \Delta_{s,t}u^{-1} \) can result in a positive word only in the cases described above, and it can result in \( r^{-1} \) concatenated with a positive word only if \( u \) is \( r\Delta_{s,t} \) itself.

(iii) Conversely, every Garside family \( S' \) of \( B^+ \) containing 1 must include \( S \). Indeed, \( S' \) must include \( \Sigma \), hence all elements \( \Delta_{s,t} \) since it is closed under right-lcm. Moreover, for \( (r, s, t) \) in \( \Sigma_3 \), the family \( S' \) must contain \( rs \) since it is closed under right-divisor and \( rs \) right-divides \( \Delta_{r,s} \); it contains \( rt \) for a similar reason and, therefore, it contains the right-lcm \( r\Delta_{s,t} \) of \( rs \) and \( rt \). So \( S' \) includes \( E \), hence \( S \) includes \( S' \), and \( S \) is the smallest Garside family containing 1 in \( B^+ \).

(iv) When no coefficient \( m_{s,t} \) is \( \infty \), \( \Sigma_1 \) and \( \Sigma_2 \) are empty and \( E \) consists of all elements \( r\Delta_{s,t} \) with \( r, s, t \) pairwise distinct atoms in \( B^+ \). As \( m_{s,t} \) does not matter, there exist \( 3(n) \) such elements for \( \Sigma \) of size \( n \).

(v) By the description of (i), \( S \) comprises 1, plus the \( n \) atoms, plus, for each of the \( \binom{n}{2} \) pairs \((s, t)\), the \( 2n - 3 \) right-divisors of \( \Delta_{s,t} \) of length \( \geq 2 \), plus, for each of the \( \binom{n}{3} \) triples \((r, s, t)\), the 3 elements \( r\Delta_{s,t}, s\Delta_{t,r}, t\Delta_{r,s} \), whence the formula. The case \( n = m = 3 \) corresponds to the affine type \( \tilde{A}_2 \), and confirms that the smallest Garside family containing 1 has 16 elements, as seen in Reference Structure 9, page 111.

Chapter X: Deligne–Lusztig varieties

Skipped proofs

(none)
Chapter XI: Left self-distributivity

Solution to selected exercises

(None)

Lemma XI.1.8.— Assume that $F$ is a partial action of a monoid $M$ on a set $X$.

(i) If the monoid $M$ is left-cancellative, then so is the category $C(M,F)$.

(ii) Conversely, if $F$ is proper and $C(M,F)$ is left-cancellative, then so is $M$.

Proof. (i) Assume that $M$ is left-cancellative and, in $C(M,F)$, the equality $x \overset{a}{\to} y \cdot y \overset{b}{\to} z = x \overset{a}{\to} y \cdot y \overset{h}{\to} z'$ holds. This implies $x \overset{gh}{\to} z = x \overset{gh}{\to} z'$, whence, by definition, $gh = gh'$ and $z = z'$, and $h = h'$ since $M$ is left-cancellative. We deduce $y \overset{h}{\to} z = y \overset{h'}{\to} z'$, and $C(M,F)$ is left-cancellative.

(ii) Conversely, assume that $F$ is proper and $C(M,F)$ is left-cancellative. Assume $gh = gh'$. As $F$ is proper, there exists $x$ in $X$ such that both $x \cdot gh$ and $x \cdot gh'$ are defined. Let $z = x \cdot gh$. By (XI.1.3), $x \cdot g$ must be defined and, putting $y = x \cdot g$, we have $z = y \cdot h = y \cdot h'$. Then, in $C(M,F)$, we have $x \overset{a}{\to} y \cdot y \overset{h}{\to} z = x \overset{a}{\to} y \cdot y \overset{h'}{\to} z'$. As $C(M,F)$ is left-cancellative, we deduce $y \overset{h}{\to} z = y \overset{h'}{\to} z'$, whence $h = h'$. Hence $M$ is left-cancellative.

Lemma XI.1.9.— Assume that $F$ is a partial action of a monoid $M$ on a set $X$.

(i) Assume that $x \cdot g$ is defined. Then $y \overset{h}{\to} - \preceq x \overset{a}{\to} - \cdot y \overset{g}{\to} -$ holds in $C(M,F)$ if and only if we have $y = x$ and $h \preceq g$ in $M$.

(ii) Assume that $x \cdot f$ and $x \cdot g$ are defined. Then $x \overset{h}{\to} -$ is a left-gcd of $x \overset{f}{\to} -$ and $x \overset{g}{\to} -$ in $C(M,F)$ if and only if $h$ is a left-gcd of $f$ and $g$ in $M$.

Proof. (i) Assume $y \overset{h}{\to} y \cdot h \cdot x \overset{g}{\to} - = (x,g,\cdot)$. Then we have $y = x$ and $hy' = g$, hence $h \preceq g$. Conversely, assume $h \preceq g$, say $hg' = g$. By (XI.1.3), the assumption that $x \cdot g$ is defined implies that $x \cdot h$ is defined, say $x \cdot h = x'$. Then we have $x \overset{h}{\to} x' \overset{g'}{\to} - = x \overset{g}{\to} -$, whence $x \overset{h}{\to} - \preceq x \overset{g}{\to} -$ in $C(M,F)$.

(ii) Assume that $x \overset{k}{\to} -$ is a left-gcd of $x \overset{f}{\to} -$ and $x \overset{g}{\to} -$ in $C(M,F)$. By (i), $h$ left-divides $f$ and $g$ in $M$. Let $h'$ be a common left-divisor of $f$ and $g$ in $M$. By (XI.1.3), the assumption that $x \cdot f$ is defined implies that $x \cdot h'$ is defined. Then $x \overset{h'}{\to} -$ is a common left-divisor of $x \overset{f}{\to} -$ and $x \overset{g}{\to} -$ in $C(M,F)$, hence it is a left-divisor of $x \overset{k}{\to} -$. By (i), this implies $h' \preceq h$. So $h$ is a left-gcd of $f$ and $g$.

Conversely, assume that $h$ is a left-gcd of $f$ and $g$. By (i), $x \overset{h}{\to} -$ left-divides $x \overset{f}{\to} -$ and $x \overset{g}{\to} -$ in $C(M,F)$. Now, consider an arbitrary common left-divisor of $\mathbb{N}_F(x,f,\cdot)$ and $x \overset{g}{\to} -$. By (i), it must be of the form $x \overset{h'}{\to} -$ with $h'$ left-dividing $f$ and $g$. Then $h$ left-divides $h$ in $M$, and $x \overset{h'}{\to} -$ left-divides $x \overset{h}{\to} -$ in $C(M,F)$. □
Lemma XI.4.16.— Assume that $M$ is a left-cancellative monoid, $F$ is a proper partial action of $M$ on some set $X$, and $(\Delta_x)_{x \in X}$ is a right-Garside map on $M$ with respect to $F$. Assume moreover that $S$ is a family of atoms that generate $M$. Now assume that $\pi : M \to M$ is a surjective homomorphism, $\pi_\#: X \to X$ is a surjective map, and, for all $x$ in $X$ and $g$ in $M$,

(XI.4.17) The value of $\pi_\#(x \cdot g)$ only depends on $\pi_\#(x)$ and $\pi(g)$;

(XI.4.18) The value of $\pi(x \cdot \Delta_x)$ only depends on $\pi_\#(x)$.

Assume finally that $\tilde{\pi} : S \to S$ is a section of $\pi$ such that, for $x$ in $X$, $s$ in $S$, and $w$ in $S^*$,

(XI.4.19) If $\pi_\#(x) \cdot s$ is defined, then so is $x \cdot \tilde{\pi}(s)$.

(XI.4.20) The relation $[w] \leq \Delta_{\pi_\#(x)}$ implies $[\tilde{\pi}(w)] \leq \Delta_x$.

Define a partial action $F$ of $M$ on $X$ by $\pi_\#(x) \cdot (g \cdot y) = \pi_\#(x \cdot g) \cdot y$, and, for $x$ in $X$, let $\Delta_x$ be the common value of $\pi(x \cdot \Delta_x)$ for $x$ satisfying $\pi_\#(x) = x$. Then $F$ is a proper partial action of $M$ on $X$ and $(\Delta_x)_{x \in X}$ is a right-Garside sequence on $M$.

Proof. First, (XI.4.17) and (XI.4.18) guarantee that the definitions of $x \cdot g$ and $\Delta_x$ are unambiguous. Next $F$ is a partial action of $M$ on $X$. Indeed, if $g$ and $h$ lie in $M$, and if $g,h$ satisfy $\pi(g) = g$ and $\pi(h) = h$, then we have $\pi(gh) = gh$ and, for $x$ in $X$ satisfying $\pi_\#(x) = \pi(x)$, we can write

$$(x \cdot g) \cdot h = \pi_\#(x \cdot g) \cdot \pi(h) = \pi_\#((x \cdot g) \cdot h) = \pi_\#(x \cdot gh) = x \cdot gh,$$

equality meaning as usual that the involved expressions are simultaneously defined and, in this case, they have the same value.

The partial action $F$ is proper. Indeed, assume that $g_1, \ldots, g_m$ are elements of $M$. As $\pi$ is surjective, there exists for every $i$ an element $y_i$ of $M$ that satisfies $\pi(y_i) = g_i$. As $F$ is proper, there exists $x$ in $X$ such that $x \cdot g_i$ is defined for each $i$. Then $\pi_\#(x) \cdot g_i$ is defined as well.

We now check that $(\Delta_x)_{x \in X}$ is a right-Garside sequence on $M$. First, let $x$ belong to $X$. There exists $x$ in $X$ satisfying $\pi_\#(x) = x$. By assumption, $x \cdot \Delta_x$ is defined, hence $x \cdot \pi(\Delta_x)$ is defined as well, and it is $\pi_\#(x \cdot \Delta_x)$. So the sequence $(\Delta_x)_{x \in X}$ satisfies (XI.1.11).

Next, the assumption that $M$ is generated by $\bigcup_{x \in X} \text{Div}(\Delta_x)$ implies that $M$ is generated by $\bigcup_{x \in X} \pi(\text{Div}(\Delta_x))$, which is $\bigcup_{x \in X} \text{Div}(\Delta_x)$. So, the sequence $(\Delta_x)_{x \in X}$ satisfies (XI.1.12).

Now, assume $x \in S$ and $s \leq \Delta_x$. Let $s$ satisfy $\pi(s) = g$ and $x$ satisfy $\pi_\#(x) = x$. By assumption, $\pi_\#(x) \cdot s$ is defined, hence, by (XI.4.19), $x \cdot \pi(s)$ is defined. As $\bigcup_s \text{Div}(\Delta_x)$ generates $M$, the element $\tilde{\pi}(g)$ is left-divisible by some non-invertible element $s$ that lies in $\text{Div}(\Delta_x)$ for some $y$. As $S$ consists of atoms, every element $s$ of $S$ belongs to $\bigcup_y \text{Div}(\Delta_y)$ and, therefore, by (XI.1.15), $s \leq \Delta_x$ must hold for every $x$ such that $x \cdot s$ is defined. Applying this to $\tilde{\pi}(g)$, we deduce $\tilde{\pi}(g) \leq \Delta_x$, whence, by (XI.1.12), $\Delta_x \leq \tilde{\pi}(\Delta_x)$, for $\tilde{\pi}(g)$. Applying $\pi$, we deduce $\Delta_x \leq \pi(\Delta_x)$. Then an easy induction on the length of $g$ shows that $\pi(s \Delta_x) \leq \tilde{\pi}(g) \Delta_x$. For every $x$ in $M$. So, the sequence $(\Delta_x)_{x \in X}$ satisfies (XI.1.13).

As of (XI.1.14), it is automatically satisfied by the sequence $(\Delta_x)_{x \in X}$ since any two elements of $M$ are supposed to admit a left-gcd.
Finally, assume that \( g \) is an element of \( M \) satisfying \( g \perp \Delta_a \) and \( y \cdot g \) is defined. Let \( x, y \) in \( X \) satisfy \( \pi_a(x) = x, \pi_a(y) = y \). Let \( \overline{w} \) be an \( A \)-word representing \( g \) in \( M \). By construction, \( \overline{w} \) left-divides \( \Delta_{a(x)} \) hence, by (XI.4.20), \( \overline{\pi^*(w)} \) left-divides \( \Delta_y \). By (XI.4.19) and an induction on the length of \( \overline{w} \), the assumption that \( y \cdot \overline{w} \) is defined implies that \( y \cdot \overline{\pi^*(w)} \) is defined. As the sequence \( \Delta_x \in X \) satisfies (XI.1.15), we deduce that \( \overline{\pi^*(w)} \) left-divides \( \Delta_y \). Applying \( \pi \), we conclude that \( w \), that is, \( g \), left-divides \( \Delta_w \). Hence, the sequence \( \Delta_x \in X \) satisfies (XI.1.15) and, therefore, it is a right-Garside sequence in \( M \). \( \square \)

**Solution to selected exercises**

**Exercice 104 (skeleton).**— Say that a set of addresses is an antichain if it does not contain two addresses, one is a prefix of the other; an antichain is called maximal if it is properly included in no antichain. (i) Show that a finite maximal antichain is a family \( \{\alpha_1, \ldots, \alpha_n\} \) such that every long enough address admits as a prefix exactly one of the addresses \( \alpha_i \). (ii) Show that, for every \( \Sigma \)-word \( w \), there exists a unique finite maximal antichain \( A_w \) such that \( T \cdot w \) is defined if and only if the skeleton of \( T \) includes \( A_w \).

**Solution.** (ii) Use induction on the length of \( w \). For \( w \) of length one, say \( w = \Sigma_\alpha \), the result is true with \( T_w = T_{\alpha} = x_1 = \{\alpha 10\} \). Assume now \( w = \Sigma_\alpha w' \). It follows from the definition of the action of \( \Sigma_\alpha \) that, for every address \( \gamma \), there exists a well defined address \( \gamma' \) such that, for every term \( T \) such that \( T \cdot \Sigma_\alpha \) is defined, the skeleton of \( T \cdot \Sigma_\alpha \) contains \( \gamma \) if and only if the skeleton of \( T \) contains \( \gamma' \). Put \( \gamma' = \Sigma_{\alpha}^{-1}(\gamma) \). Then we claim that the result holds with \( A_w = \{\alpha 10\} \cup \Sigma_{\alpha}^{-1}(A_{w'}) \). Indeed, \( T \cdot w \) is defined if and only if \( T \cdot \Sigma_\alpha \) and \( (T \cdot \Sigma_\alpha) \cdot w' \) are defined, hence, by induction hypothesis, if and only if the skeleton of \( T \) contains \( \alpha 10 \) and the skeleton of \( T \cdot \Sigma_\alpha \) includes \( A_{w'} \).

**Exercice 105 (preservation).**— Assume that \( M \) is a left-cancellative monoid and \( F \) is a partial action of \( M \) on a set \( X \). (i) Show that, if \( M \) admits right lcms (resp. conditional right-lcms), then so does the category \( \mathcal{GF}(M,X) \). (ii) Show that, if \( M \) is right-Noetherian, then so is \( \mathcal{GF}(M,X) \).

**Solution.** By definition, \( (x,f,y) \in (x',f',y') \) in \( \mathcal{GF}(M,X) \) implies \( x' = x \) and \( f \perp f' \) in \( M \). So the assumption that \( M \) is right-Noetherian implies that \( \mathcal{GF}(M,X) \) is right-Noetherian as well. Assume that \( (x,f,y) \) and \( (x,g,z) \) admit a common right-multiple in \( \mathcal{GF}(M,X) \), say \( (x,f,y)(g',f') = (x,g,z)(z',f') \). Then \( f'g' = f' \) holds in \( M \). As \( M \) is a right-lcm monoid, \( f \) and \( g \) admit a right-lcm \( h \), and we have \( f \leq c, g \leq h, \) and \( h \leq f'g' \). By assumption, \( x \cdot f'g' \) is defined, hence so is \( x \cdot h \), and it is obvious to check that \( (x,h,x \cdot h) \) is a right-lcm of \( (x,f,y) \) and \( (x,g,z) \) in \( \mathcal{GF}(M,X) \).

**Exercice 107 (Noetherianity).**— Assume that \( F \) is a proper partial action of a monoid \( M \) on a set \( X \) and there exists a map \( \mu : X \to \mathbb{N} \) such that \( \mu(x \cdot g) > \mu(x) \) holds whenever \( g \) is not invertible. Show that \( M \) is Noetherian and every element of \( M \) has a finite height.

**Solution.** Let \( g \) be a non-invertible element of \( M \), and let \( g_1 \cdots g_p \) be any decomposition of \( g \). At the expense of possibly gathering entries, we may assume that each element \( g_i \) is non-invertible. As \( F \) is proper, there exists \( x \) in \( X \) such that
Exercice 109 (common multiple).— Assume that $M$ is a left-cancellative monoid, $F$ is a partial action of $M$ on $X$, and $(\Delta_x)_{x \in X}$ is a right-Garside sequence in $M$ with respect to $F$. Show that, for every $x$ in $X$, any two elements of $\text{Def}(x)$ admit a common right-multiple that lies in $\text{Def}(x)$.

Solution. Let $f, g$ belong to $\text{Def}(x)$. Then $(x, f, x \cdot f)$ and $(x, g, x \cdot g)$ are two elements of $\mathcal{C}_F(M, X)$ sharing the same source. By Proposition XI.3.25, $\mathcal{C}_F(M, X)$ possesses a right-Garside map, hence any two elements of $\mathcal{C}_F(M, X)$ with the same source admit a common right-multiple. So do in particular $(x, f, x \cdot f)$ and $(x, g, x \cdot g)$. A common right-multiple must be of the form $(x, h, x \cdot h)$ where $h$ is a common right-multiple of $f$ and $g$ in $M$. Then $h$ lies in $\text{Def}(x)$.

Chapter XII: Ordered groups

Skipped proofs

(none)

Solution to selected exercises

Exercice 111 (braid ordering).— Show that $1 <_\sigma \sigma_1\sigma_2 \leq_{\sigma_p} (\sigma_1\sigma_2\sigma_1)^{2p} \sigma_2^2$ holds in $B_3$ for $p > 0$.

Solution. Put $g = (\sigma_1\sigma_2\sigma_1)^{2p} \sigma_2^2$ and $g' = \sigma_1\sigma_2$. We find $g'^{-1}g = \sigma_1(\sigma_1\sigma_2\sigma_1)^{2p-1} \sigma_2^2$. Hence $g' <_\sigma g$ is true.

Exercice 112 (limit of conjugates).— Assuming that $<_\sigma$ is a limit of its conjugates in $B_3$, show the same result in $B_n$. [Hint: Use the subgroup of $B_n$ generated by $\sigma_{n-2}$ and $\sigma_{n-1}$, which is isomorphic to $B_3$.]

Solution. Let $P_n$ be the positive cone of $<_\sigma$ on $B_n$, and let $H$ be the subgroup of $B_n$ generated by $\sigma_{n-2}$ and $\sigma_{n-1}$. Then $H$ is isomorphic to $B_3$ and the $<_\sigma$-ordering of $B_n$ restricted to $H$ corresponds with the $<_\sigma$-ordering of $B_3$. Moreover, the positive cone for the $<_\sigma$-ordering of $H$ is $P_n \cap H$. Let $S$ be a finite subset of $P_n$. By assumption, there exists $f$ in $H$ such that $f^{-1}hf$ is lies in $P_n$ for every $h$ in $S \cap H$, and $f(P_n \cap H)f^{-1}$ is distinct from $P_n \cap H$. Assume now $h \in S \setminus H$. By assumption, $h$ is $\sigma_i$-positive for some $i < n - 2$. Its conjugate $f^{-1}hf$ is $\sigma_i$-positive as well, hence it lies in $P_n$. We deduce $f^{-1}Sf \subseteq P_n$, hence $S \subseteq fP_nf^{-1}$. Finally, $fP_nf^{-1}$ and $P_n$ are distinct, because their intersections with $H$ are distinct.

Exercice 113 (closure of conjugates).— Let $P_n$ be the positive cone of the ordering $<_\sigma$ on $B_n$ considered in Example XII.1.23. Show that the closure of the conjugates of $P_n$ in $\text{LO}(B_n)$ is a Cantor set.
Solution. Let $Z_n$ be the family of all conjugates of $P_n$. We saw in Example XII.1.23 that $P_n$ is a limit of its conjugates, hence it is a limit point in $Z_n$. Hence so is every conjugate of $P_n$, that is, every point of $Z_n$ is a limit point in $Z_n$. By continuity, every point in the closure of $Z_n$ is a limit point. Next, the closure of $Z_n$ is a closed subspace in a totally disconnected nonempty compact metric space, hence it is itself totally a disconnected, compact, and metric space. As it is nonempty and every point is a limit point, the closure of $Z_n$ is homeomorphic to the Cantor set.

Exercise 114 (space $\text{LO}(B_\infty)$).— Show that every point in the space $\text{LO}(B_\infty)$ is a limit of its conjugates and that $\text{LO}(B_\infty)$ is homeomorphic to the Cantor set (contrary to the spaces $\text{LO}(B_n)$ for finite $n$).

Solution. Consider an arbitrary positive cone $P$ for a left-invariant ordering of $B_\infty$ and suppose $S$ is a finite subset of $P$. We will show there is a positive cone $\sigma_i P \sigma_i^{-1}$ in $B_\infty$ that also includes $S$ and is distinct from $P$. Choose $n$ such that $S$ is included in $B_n$. Then, for each $i > n$, every braid in $S$ commutes with $\sigma_i$, so we have $S = \sigma_i S \sigma_i^{-1} \subseteq \sigma_i P \sigma_i^{-1}$. On the other hand, there exists $i > n$ such that the sets $P$ and $\sigma_i P \sigma_i^{-1}$ are different. For otherwise, using $\text{sh}$ for the shift endomorphism that maps $\sigma_i$ to $\sigma_{i+1}$ for every $i$, consider the subgroup $\text{sh}^n(B_\infty)$, which is isomorphic to $B_\infty$. The sets $P \cap \text{sh}^n(B_\infty)$ and $(\sigma_i P \sigma_i^{-1}) \cap \text{sh}^n(B_\infty)$ are positive cones for orderings of $\text{sh}^n(B_\infty)$. If $P = \sigma_i P \sigma_i^{-1}$ is true for each $i > n$, then the cone $P \cap \text{sh}^n(B_\infty)$ of $\text{sh}^n(B_\infty)$ is invariant under conjugation by all elements of $\text{sh}^n(B_\infty)$. This would imply that $\text{sh}^n(B_\infty)$ and, therefore, $B_\infty$ are bi-orderable, which is not true.

Exercise 116 (non-terminating reversing).— Assume that $(S, R)$ is a triangular presentation. (i) Show that, if a relation of $\tilde{R}$ has the form $s = w$ with $\log(w) > 1$ and $w$ finishing with $s$, then the monoid $(S \mid R)^+$ is not of right-O-type. (ii) Let $(S, R)$ be the maximal right-triangular deduced from $(S, R)$. Show that, if a relation of $\tilde{R}$ has the form $s = w$ with $w$ beginning with $(uv)^r us$ with $r \geq 1$, $u$ nonempty, and $v$ such that $v^{-1}s$ reverses to a word beginning with $s$, hence in particular if $v$ is empty or it can be decomposed as $u_1, \ldots, u_m$ where $u_ks$ is a prefix of $w$ for every $k$, then $s^{-1}us$ cannot be terminating, and deduce that $(S \mid R)^+$ is not of right-O-type. (iii) Show that a relation $a = bab^2a^2\ldots$ is impossible in a right-triangular presentation for a monoid of right-O-type.

Solution. (i) If $R$ contains a relation $s = us$ with $u$ nonempty, $s = [u]^+$ holds in $(S \mid R)^+$, whereas $1 = [w]^+$ fails. So $(S \mid R)^+$ is not right-cancellative. (ii) Write the involved relation $s = (uv)^r usw_1$ with $v^{-1}s \cap_{\tilde{R}} sw_2$. We find
\[
s^{-1}us \cap_{\tilde{R}} w_1^{-1}s^{-1}(vu)^{-(r-1)}u^{-1}s^{-1}v_1^{-1}s
\cap_{\tilde{R}} w_1^{-1}s^{-1}(vu)^{-(r-1)}u_{1}^{-1}sw_2 \cap_{\tilde{R}} w_1^{-1}s^{-1}(vu)^{-(r-1)}(vu)^{-1}vusw_1w_2
\cap_{\tilde{R}} w_1^{-1}s^{-1}vusw_1w_2 \cap_{\tilde{R}} w_1^{-1}w_2^{-1} \cdot s^{-1}us \cdot w_1w_2.
\]
We deduce that $s^{-1}us \cap_{\tilde{R}} (w_1^{-1}w_2^{-1})^n \cdot s^{-1}us \cdot (w_1w_2)^n$ holds for every $n$, and therefore, it is impossible that $s^{-1}us$ leads in finitely many steps to a positive-negative word. Proposition II.4.51 (completeness), right-reversing is complete for
(S, R), the elements s and [u]s admit no common right-multiple in ⟨S | R⟩⁺, and the latter cannot be of right-Ω-type.

(iii) The right-hand side of the relation can be written as (ba)babb(2ba)a..., eligible for (ii) with u = ba and v = bab · b, a product of two words u₁, u₂ such that u₁a is a prefix of the right-hand term of the relation.

**Exercice 117 (roots of Garside element).** Assume that M is a left-cancellative monoid generated by a set S (i) Show that, for δ, g in a left-cancellative monoid M, a necessary and sufficient condition for δ to right-dominate g is that there exist m ≥ 1 satisfying (∗)∀k≥0 (gδkm+m−1 ≼ δkm+1). (ii) Assume that δ⁰ is a right-Garside element in M. Show that δ right-dominates every element g that satisfies gδ⁻⁰ ≼ δ.

**Solution.** (i) If δ right-dominates g, then, by definition, (∗) holds with m = 1. Conversely, assume (∗). Let n be a nonnegative integer. Let k be maximal with km ≤ n. Then we have n ≤ km + m − 1, and (∗) implies gδⁿ ≼ gδkm+m−1 ≼ δkm+1 ≼ δⁿ+1, so δ right-dominates s.

(ii) Put ∆ = δ⁰, and let φ be the (necessarily unique) endomorphism of M witnessing that ∆ is right-quasi-central. First, we have δΔ = δ⁰+1 = Δφ(δ), whence φ(δ) = δ since M is left-cancellative. Next, we claim that g ≼ h implies φ(g) ≼ φ(h). Indeed, by definition, g ≼ h implies the existence of h′ satisfying gh′ = h, whence φ(g)φ(h′) = φ(h) since φ is an endomorphism. This shows that φ(g) ≼ φ(h) is satisfied. So, in particular, and owing to the above equality, g ≼ δ implies φ(g) ≼ δ. Now assume gδ⁻ᵐ ≼ δ. Then, for every k, we find

\[ gδkm+m−1 = gδⁿ−1 \Delta^k = \Delta^kδ^k(gδm−1) ≼ Δ^kφ^k(δ) = Δ^kδ = δkm+1, \]

and we conclude that δ right-dominates g by (i).

**Exercice 119 (right-ceiling).** Assume that M is a cancellative monoid of right-O-type, and that s₁...s₁ is a right-top S-word in M such that [s₁...s₁]⁺ is central in M. Show that s₁ = s₁ must hold for every i, and deduce that s₁...s₁ is the right-S-ceiling in M.

**Solution.** Let ∆ = s₁...s₁. First, we have g ≼ ∆ for every g in S⁺, so that ∆ right-dominates S⁺. Hence, by Lemma XII.3.9, the right-S-ceiling is periodic with period s₁...s₁. Now consider its length ℓ + 1 final fragment s₁s₁...s₁. Then, in M, we have s₁s₁...s₁ = s₁s₁...s₁, so s₁s₁...s₁ and s₁s₁...s₁ are two right-top S-words of length ℓ + 1. By uniqueness of the right-S-ceiling, these words must coincide, which is possible only for s₁ = ... = s₁.

**Exercice 123 (no triangular presentation).** Assume that M is a monoid of right-O-type that is generated by a, b, c with a ≻ b ≻ c and b, c satisfying some relation b = cv with no a in v. (i) Prove that, unless M is generated by b and c, there is no way to complete b = cv with a relation a = bu so as to obtain a presentation of M. (ii) Deduce that no right-triangular presentation made of b = cbc (Klein bottle relation) or b = cb²c (Dubrovina–Dubrovin braid relation) plus a relation of the form a = b... may define a monoid of right-O-type.

**Solution.** For a contradiction, assume that (a, b, c; a = bu, b = cv) is a presentation of M. If there is no a in u, the assumption that a = bu is valid in M implies that a belongs to the submonoid generated by b and c, so M must be generated
by $b$ and $c$. Assume now that there is at least one $a$ in $u$. As $a$ does not occur in $b = cv$, a word containing $a$ cannot be equivalent to a word not containing $a$. This implies that $a$ is preponderant in $\{a, b, c\}$. Indeed, assume that $g, h$ belong to the submonoid of $M$ generated by $b$ and $c$. By the above remark, $hag' = g$ is impossible, hence so is $ha \leq g$. As, by assumption, $M$ is of right-$O$-type, we deduce $g \leq ha$. Then Proposition XII.3.14 gives the result.

**Exercice 124 (Birman–Ko–Lee generators).**— Put $b_{i,j} = a_{i,j}^{(-1)^{i+1}}$ in the braid group $B_n$. Show that, for every $n$, the monoid $B_n^\oplus$ is generated by the elements $b_{i,j}$. Solution. By Lemma XII.3.13, an element of $B_n$ belongs to $B_n^\oplus$ if and only if it is either $\sigma_i$-positive for some odd $i$ or $\sigma_i$-negative for some even $i$. The braid relations imply $a_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_{i} \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ for $i < j$, hence $a_{i,j}$ is $\sigma_i$-positive, and $b_{i,j}$ is $\sigma_i$-positive for odd $i$ and $\sigma_i$-negative for even $i$. Therefore, $b_{i,j}$ belongs to $B_n^\oplus$ for all $i, j$. Conversely, in $B_n$, we have

$$\sigma_i \cdots \sigma_{n-1} = (\sigma_i \cdots \sigma_{n-2} \sigma_{n-1}^{-1}) (\sigma_i \cdots \sigma_{n-2} \sigma_{n-3}^{-1} \cdots \sigma_{i+1}^{-1}) \cdots (\sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{1}^{-1})(\sigma_i),$$

whence $s_i = (\sigma_i \cdots \sigma_{n-1})^{(-1)^{i+1}} = b_{i,n} b_{n,i} \cdots b_{i,i+1}$ for odd $i$, and $b_{i,i} \cdots b_{i,n-1} b_{i,n}$ for even $i$. Hence $s_i$ belongs to the submonoid of $B_n$ generated by the $b_{i,j}$’s and, finally, $B_n^\oplus$ coincides with the latter.

**Chapter XIII: Set-theoretic solutions of YBE**

**Skipped proofs**

**Solution to selected exercises**

**Exercice 125 (bijective RC-quasigroup).**— Assume that $(X, \ast)$ is an RC-quasigroup. Let $\psi : X \to X$ and $\Psi : X \times X \to X \times X$ be defined by $\psi(a) = a \ast a$ and $\Psi(a, b) = (a \ast b, b \ast a)$. Show that $\Psi$ is injective (resp. bijective) if and only if $\psi$ is.

Solution. If $\Psi$ is injective, then so is $\psi$ since $\psi(a) = \psi(a')$ implies $\Psi(a, a) = \Psi(a', a')$. On the other hand, if $\Psi$ is surjective, then so is $\psi$: for every $c$, there exists $(a, b)$ satisfying $\Psi(a, b) = (c, c)$. As seen in the proof of Lemma XII.2.23(iii), this implies $a = b$, whence $\psi(a) = c$. For the converse, we first compute $\psi(b \ast a) = (b \ast a) \ast (b \ast a) = (a \ast b) \ast (a \ast a) = (a \ast b) \ast \psi(a)$. Assume that $\psi$ is injective and $\Psi(a, b) = \Psi(a', b') = (c, d)$ holds. By $(\ast)$, we have $c \ast \psi(a) = \psi(d) = c \ast \psi(a')$. As the left-translation by $c$ and $\psi$ are injective, we deduce $\psi(a) = \psi(a')$, whence $a = a'$, and a symmetric argument gives $b = b'$, so $\Psi$ is injective. Finally, assume that $\psi$ is bijective and $c, d$ belong to $X$. As the left-translation by $c$ and $\psi$ are surjective, we can find $a$ satisfying $c \ast \psi(a) = \psi(d)$, whence, by $(\ast)$, $\psi(b \ast a) = \psi(d)$ and, similarly, we can find $b$ satisfying $\psi(a \ast b) = \psi(c)$. As $\psi$ is injective, we deduce $a \ast b = c$ and $b \ast a = d$, whence $\Psi(a, b) = (c, d)$. So $\Psi$ is surjective, hence bijective.
Exercise 128 (right-complement).— Assume that $(X, \ast)$ is an RC-quasigroup and $M$ is the associated structure monoid. (i) Show that, for every element $f$ in $M \cap X^p$, the function from $X$ to $X \cup \{1\}$ that maps $t$ to $f \setminus t$ takes pairwise distinct values in $X$ plus at most $p$ times the value 1. (ii) Deduce that, for $I$ a finite subset of $X$ with cardinal $n$, the right-lcm $\Delta_I$ of $I$ lies in $X^n$.

Solution. (i) We use induction on $p$. For $p = 0$, that is, for $f = 1$, we have $f \setminus t = t$, which takes pairwise distinct values in $X$. For $p = 1$, that is, for $f$ in $X$, we have $f \setminus f = 1$ and $f \setminus t = f \ast t$ for $t \neq f$. As $t \mapsto f \ast t$ injective, the expected result is true. Assume $p \geq 2$ and write $f = g s$ with $g$ in $X^{p-1}$. Then, by the formula for an iterated right-complement, we have $f \setminus t = s \setminus (g \setminus t)$, whence $f \setminus t = 1$ if $g \setminus t$ lies in $\{1, s\}$ and $f \setminus t = s \ast (g \setminus t)$ otherwise. Then the result follows from the induction hypothesis.

(ii) Use induction on $n \geq 0$. For $n = 0$, we have $\Delta_I = 1$. For $n = 1$, say $I = \{s\}$ with $s$ in $X$, we have $\Delta_I = s$, an element of $X$. For $n = 2$, say $I = \{s, t\}$ with $s \neq t$, we have $\Delta_I = s(s \ast t)$, so the induction starts. Assume $n \geq 3$ and $I = \{s_1, \ldots, s_n\}$. Let $J = \{s_1, \ldots, s_{n-2}\}$. By induction hypothesis, $\Delta_J$ belongs to $X^{n-2}$, and $\Delta_{J \cup \{s_{n-1}\}}$, which is $\Delta_J(\Delta_J \setminus s_{n-1})$, belongs to $X^{n-1}$. This implies $\Delta_J \setminus s_{n-1} \neq 1$, so $\Delta_J \setminus s_{n-1}$ must lie in $X$. For the same reason, $\Delta_J \setminus s_n$ lies in $X$. Moreover, as $s_{n-1}$ and $s_n$ are distinct, we have $\Delta_J \setminus s_{n-1} \neq \Delta_J \setminus s_n$ by (i). Now, as $I = J \cup \{s_{n-1}\} \cup \{s_n\}$, Proposition II.2.12 (iterated lcm) gives $\Delta_I = \Delta_J \cup (\Delta_J \setminus s_{n-1})(\Delta_J \setminus s_n)$ with a commutative diagram as below, and the question is to know whether the last term may be 1.

$$\begin{array}{c|c|c}
\Delta_I & \Delta_J \setminus s_{n-1} \neq 1 & (\Delta_J \setminus s_{n-1})(\Delta_J \setminus s_n) \\
\downarrow & \downarrow & \downarrow \\
\Delta_J \setminus s_n \neq 1 & (\Delta_J \setminus s_{n-1}) \setminus (\Delta_J \setminus s_n) \in X? \\
\end{array}$$

Now we saw above that $\Delta_J \setminus s_{n-1}$ and $\Delta_J \setminus s_n$ are distinct elements of $X$, so the element $(\Delta_J \setminus s_{n-1}) \setminus (\Delta_J \setminus s_n)$ is equal to $(\Delta_J \setminus s_{n-1}) \ast (\Delta_J \setminus s_n)$, an element of $X$. Hence $\Delta_I$ belongs to $X^n$, as expected.

Exercise 128 (I-structure).— Assume that $(X, \ast)$ is a bijective RC-quasigroup.

(i) Show (by a direct argument) that the map $\pi : X^* \times X$ to $X$ defined by $1_{\pi} = 1$, $s_{\pi} = s$ for $s$ in $X$, and $(u \ast v)_{\pi} = v_{\pi}(u_{\pi})$ induces a well-defined map from $M \times X$ to $X$. (ii) Show that the map $\nu$ from $X^*$ to $M$ defined by $\nu(1) = 1$, $\nu(s) = s$, and $\nu(us) = \nu(u) \cdot (\nu(v) \ast s)$ for $s$ in $X$ induces a well-defined map from $\mathbb{N}^{(X)}$ to $M$.

Solution. (i) Owing to the presentation of $M$, it suffices to check that the translations associated with $r((r \ast s)$ and $s((s \ast r)$ coincide. Now, using the RC-law, we find $(r_{\ast}(r \ast s))_{\pi} = (r \ast s)_{\pi}((r \ast t) = (s \ast r)_{\ast}(s \ast t) = (s_{\ast}(s \ast t))_{\pi}$. (ii) As $\mathbb{N}^{(X)}$ is the quotient of $X^*$ by the equivalence relation generated by all pairs $(u|s|t|v,u|t|s|v)$ with $s, t$ in $X$, it suffices to check that the images of such words coincide. Now we find

$$\nu(u|s|t|v) = \nu(u) \cdot (\nu(u|s|t|v) \ast (\nu(v) \ast s)) = \nu(u) \cdot \nu(u) \cdot \nu(v) \ast s$$

and

$$\nu(u|t|s|v) = \nu(u) \cdot (\nu(v) \ast t) \ast (\nu(v) \ast s)$$

whence the equality. Then multiplying by $v$ on the right gives the same result.
Exercise 129 (parabolic submonoid).— (i) Assume that \((X, \rho)\) is a finite involutive nondegenerate set-theoretic solution of YBE and \(M\) is the associated structure monoid. Show that a submonoid \(M_1\) of \(M\) is parabolic if and only if there exists a (unique) subset \(I\) of \(X\) satisfying \(\rho(I \times I) = I \times I\) such that \(M_1\) is the submonoid of \(M\) generated by \(I\). (ii) Assume that \((X, \star)\) is a finite RC-system and \(M\) is the associated structure monoid. Then a submonoid \(M_1\) of \(M\) is parabolic if and only if there exists a (unique) subset \(I\) of \(X\) such that \(I\) is closed under \(\star\) and \(M_1\) is the submonoid generated by \(I\). (iii) Show that, if \((X, \star)\) is an infinite RC-quasigroup, there may exist subsets \(I\) of \(X\) that are closed under \(\star\) but the induced RC-system \((I, \star)\) is not an RC-quasigroup.

Solution. (i) Assume that \(M_1\) is a parabolic submonoid of \(M\). Set \(I = M_1 \cap X\). Since \(X\) generates \(M\) and \(M_1\) is closed under factors, \(M_1\) is generated by \(I\). Now, assume that \(a, b\) belong to \(I\) and \((c, d) = \rho(a, b)\) holds. Then \(ab\) belongs to \(M_1\), and \(ab = cd\) holds in \(M\). As \(M_1\) is closed under factor, we deduce that \(c\) and \(d\) lie in \(I\), whence \(\rho(I \times I) \subseteq I \times I\). As \(\rho\) is involutive, the latter inclusion must be an equality. Conversely assume \(I\) is a subset of \(X\) satisfying \(\rho(I \times I) = I \times I\) and \(M_1\) is the submonoid of \(M\) generated by \(I\). The monoid is (right)-Noetherian, so, in order to prove that \(M_1\) is a parabolic submonoid, and owing to Proposition VII.1.32 (parabolic subcategory), it is sufficient to establish that \(M_1\) is closed under factor and right-comultiple, that is, in a context where right-lcms exist, that \(M_1\) is closed under factor and right-lcm. Now, an obvious induction shows that every \(X\)-word that is equivalent to an \(I\)-word is itself an \(I\)-word, implying that \(M_1\) is closed under factor. On the other hand, let \(\rho_1\) be the restriction of \(\rho\) to \(I \times I\). The assumption that \((X, \rho)\) is a set-theoretic solution of YBE implies that so is \((I, \rho_1)\), and the assumption that \((X, \rho)\) is involutive implies that so is \((I, \rho_1)\). Finally, the assumption that \((X, \rho)\) is nondegenerate implies that so is \((I, \rho_1)\): with the notation of Definition XIII.1.8, the left-translations associated with \(\rho_1\) and right-translations associated with \(\rho_2\) are injective on \(X\), hence so are their restrictions to \(I\), and therefore, the latter are bijective since \(I\) is finite. Now, let \(a, b\) belong to \(I\). As \((I, \rho_1)\) is an involutive nondegenerate solution of YBE, there exist \(c, d\) in \(I\) satisfying \(ac = bd\) in \(M\), hence in \(M_1\), and \(M_1\) is closed under right-lcm in \(M\). Hence \(M_1\) is a parabolic submonoid of \(M\).

(ii) Translate the result of (i) in terms of the operation \(\star\).

(iii) Consider \(X = \mathbb{Z}\) and \(x \star y = y + 1\). Then the restriction of \(\star\) to \(\mathbb{N}\) does not give an RC-quasigroup.

Chapter XIV: More examples

Skipped proofs

(None)

Solution to selected exercises

(None)
Addenda to “Foundations of Garside Theory”, by Dehornoy et al.

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