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**Singularité des séries de Dirichlet associées à des polynômes de plusieurs variables et applications en théorie analytique des nombres. (French. English summary) [Singularity of the Dirichlet series associated with multivariate polynomials and applications in analytic number theory]**

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Let  $P \in \mathbf{R}[X_1, X_2, \dots, X_n]$  be a polynomial in many variables. The function  $Z(P; s)$  of a complex variable  $s$  is called the Dirichlet series associated with  $P$  if  $Z(P; s) = \sum_{m \in (\mathbf{N}^*)^n} 1/P(m)^s$ .

The investigation of Dirichlet series has a fundamental place in number theory. Many arithmetical problems can be formulated in terms of singularities of such series. The Dirichlet series  $Z(P; s)$  were introduced by R. H. Mellin [*Acta Soc. Sci. Fennicae* **29** (1900), no. 4, 1–49] with  $P \in \mathbf{C}[X_1, \dots, X_n]$ .

Systematic investigations of such series were carried out by P. Cassou-Noguès [*Ann. Inst. Fourier (Grenoble)* **31** (1981), no. 4, vii, 1–35; [MR 83e:12011](#); in *Journées Arithmétiques (Metz, 1981)*, 1–15, Astérisque, 94, Soc. Math. France, Paris, 1982; see [MR 84d:10005](#)], P. Sargos [*Ann. Inst. Fourier (Grenoble)* **34** (1984), no. 3, 83–123; [MR 86m:32004](#); “Séries de Dirichlet associées à des polynômes de plusieurs variables”, Thèse d’Etat, Univ. Bordeaux I, 1987; per bibl.; *J. Reine Angew. Math.* **367** (1986), 139–154; [MR 87j:11087](#)] and B. Lichtin [*Compositio Math.* **65** (1988), no. 1, 81–120; [MR 89d:32030](#)].

The principal problems concerning the Dirichlet series  $s \rightarrow Z(P; s)$  are: (1) How do domains  $D$  of  $\mathbf{C}$  look for which the function  $Z(P; \cdot)$  exists and is holomorphic? (2) Existence of the meromorphic extension to  $\mathbf{C}$ , determination of poles and majorization of orders of poles. (3) Majorization of the meromorphic extension in the vertical domains  $\{s \in \mathbf{C} : a < \operatorname{Re}(s) < b\}$ ;  $a, b \in \mathbf{R}$ ,  $a < b$ . (4) The Riemann type hypothesis for  $Z(P; s)$  (this problem, however, is not considered in the paper).

The above-mentioned authors observed that ellipticity and hypoellipticity of  $P$  are needed

for  $Z(P; s)$  to be holomorphic and have meromorphic extension. The author introduces two new hypoelliptic type conditions ( $H_0$  and  $H_0S$ ) which generalize those considered by the above-mentioned authors. The main results of the paper are given in the following four theorems.

Theorem 1. Under the assumption  $H_0$  on  $P \in \mathbf{R}[X_1, \dots, X_n]$ ,  $Z(P; s)$  exists and is holomorphic in the half-plane  $\{s: \operatorname{Re}(s) > \alpha\}$ . The smallest such possible  $\alpha$  is denoted by  $\sigma_0(P)$  and is called the abscissa of convergence of  $Z(P, s)$ .

Theorem 2. If  $H_0S$  holds for  $P \in \mathbf{R}[X_1, \dots, X_n]$  (let  $\sigma_0(P)$  be the coordinate of convergence from Theorem 1) then the function  $s \rightarrow Z(P; s)$ , which is holomorphic in the half-plane  $\{s: \operatorname{Re}(s) > \sigma_0(P)\}$ , has a meromorphic extension to  $\mathbf{C}$  with poles of order  $n$  which are contained in a sequence of the form  $\sigma_0(P) - k/M$ ,  $k \in \mathbf{N}$ . Here  $M = M(P)$  does not depend on  $P$ .

Theorem 3. Let  $P \in \mathbf{R}[X_1, \dots, X_n]$  satisfy  $H_0S$ . Then the above-mentioned meromorphic extension  $\widehat{Z}(P; s)$  of  $Z(P; s)$  has the following approximation: there exists  $A = A(P) > 0$  such that for each  $\varepsilon > 0$ ,  $N > 0$  and  $\delta > 0$ ,  $\widehat{Z}(P; s) \ll_{N, \varepsilon, \delta} 1 + |t|^{A(\sigma_0 - \sigma) + \varepsilon}$  if  $\operatorname{Re}(s) \geq \sigma_0 - N$  and  $t = \operatorname{Im}(s)$  is such that  $|t| \geq \delta$ .

The proofs are technically very complicated and the author uses many advanced techniques. For example, in some considerations he uses the deep theorem of Hironaka on the resolution of singularities.

**Reviewed** by [Andrzej Mądrecki](#)

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