The isotopy problem of braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen, France

N-KOOK Seminar, Osaka State University, May 16, 2015

- The braid isotopy problem is a problem of medium difficulty, with many (really) different solutions illustrating various approaches to Artin's braid groups.
- Here: a survey of some solutions:
  - one algebraic solution: the greedy normal form
  - two topological solutions: Dynnikov's coordinates, Bressaud's relaxation method
    [and two more: the alternating normal form (yesterday), handle reduction (ILDT)]

Plan:

- 1. The braid isotopy problem
- 2. Greedy normal form and the Garside structure
- 3. Dynnikov's coordinates
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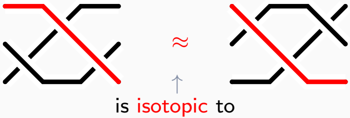
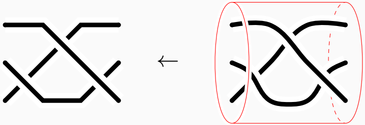
- A 3-strand braid diagram:

(no U-turn allowed)

- Isotopy Problem:
  Given two n-strand braid diagrams, can one deform them to one another?

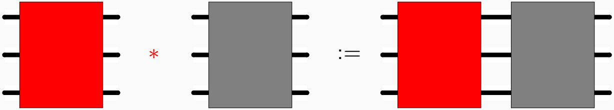


- More formally: view braid diagrams as projections of 3D-diagrams in $D^2 \times (0,1)$,

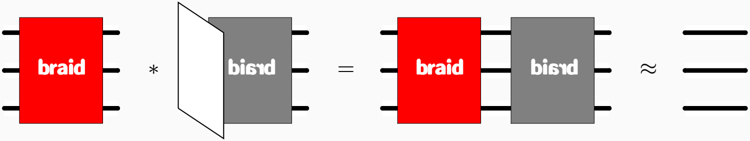


and consider ambient isotopy leaving the end-disks fixed.

- Concatenation of braid diagrams:



  - Associative;
  - Compatible with isotopy, hence induces a well-defined product on classes;
  - Admits the unbraided diagram $[\emptyset]$ as a neutral element;
  - Every diagram has an inverse, its mirror-image:

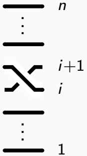


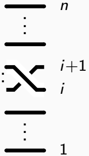
- For every $n \geqslant 1$: the group B_n of n-strand braids.

↑ isotopy class of braid diagrams

- The group structure of B_n makes the Braid Isotopy Problem easier:
  - Reduces to the Braid Triviality Problem: $D' \approx D \;\Leftrightarrow\; D^{-1} * D' \approx [\emptyset]$.
  - Enables one to use algebraic tools, provided one has a presentation of B_n.

- Artin generators: Every n-strand braid diagram is a (finite) concatenation of elementary diagrams with one crossing, hence of the form

σ_i : 

or σ_i^{-1} : 

with $1 \leqslant i < n$.

- <u>Theorem</u> (Artin, 1926): The group B_n admits the presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

  - Proof: Isotopy of piecewise linear diagrams is generated by Δ-moves. □

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- Braid Isotopy reduced to the Word Problem for B_n with respect to $\{\sigma_1, ..., \sigma_{n-1}\}$:

  given a braid word w, decide whether w represents 1 in B_n.

  ↑

  a word in the letters $\sigma_1^{\pm 1}, ... \sigma_{n-1}^{\pm 1}$.

- (Novikov, 1952) There exists a finitely presented group with an unsolvable Word Problem.

- Here: (Garside) Use the monoid.

- Theorem (Garside, 1969): *Let B_n^+ be the monoid with presentation*

$$\left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle^+.$$

Then B_n^+ embeds in B_n and B_n is a group of fractions for B_n^+.

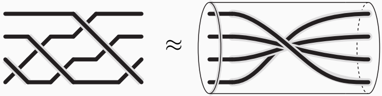
↑

every element of B_n can be written $\beta^{-1}\gamma$ with $\beta, \gamma \in B_n^+$

- Proof: Show that B_n^+ is cancellative and admits common multiples. □

- An effective way of reducing from B_n to B_n^+:

- Lemma (Garside): *Inductively define* Δ_n *by* $\Delta_1 = 1$, $\Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1$.



Then, for every (signed) n-strand braid word w, one can find $p \geqslant 0$ *and a positive n-strand braid word* w' *and satisfying* $\Delta_n^p w \equiv w'$.

- Then: $w \equiv \varepsilon \quad \Leftrightarrow \quad w' \equiv \Delta_n^p \quad \Leftrightarrow \quad w' \equiv^+ \Delta_n^p$

  ε ↑ the empty word

  $\equiv$ ↑ equivalence generated by braid relations and $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$

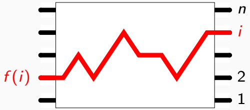
  $\equiv^+$ ↑ equivalence generated by braid relations alone

- Now: $\equiv^+$ is decidable, as it preserves word-length.

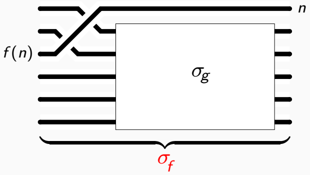
- Hence: A (theoretical) solution to the Braid Isotopy Problem: starting from w,
  - 1. find p and w' positive satisfying $\Delta_n^p w \equiv w'$;
  - 2. test $w' \equiv^+ \Delta_n^p$ by systematically enumerating the $\equiv^+$-class of w'.

- To improve the previous solution and make it tractable:
  define (efficiently computable) normal forms on B_n^+.

- Every n-strand braid gives a permutation of $\{1, ..., n\}$:
  follow the positions of the strands:
  - short exact sequence
    $$1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$



- Inductively define a (set-theoretic) section for the projection of B_n onto $\mathfrak{S}_n$:
  for $f = (n, f(n)) \circ g$ with $g \in \mathfrak{S}_{n-1}$,
  put $\sigma_f := \sigma_{f(n)} \cdots \sigma_{n-1}\sigma_g$
  - a family of $n!$ permutation braids in B_n^+.



- Lemma: *Permutations braids are the (left- and right-) divisors of Δ_n in B_n^+.*

  ↑ β left-divides γ if $\exists\gamma'$ $(\beta\gamma' = \gamma)$.

• Theorem (Garside 1969): *With respect to (left- and right-) divisibility, B_n^+ is a lattice.*

↑ least common multiples and greatest common divisors exist

• Corollary: *For every positive n-strand braid β, there exists a unique maximal permutation braid left-dividing β.*

↑ namely: the left-gcd of β and Δ_n

▶ A distinguished decomposition:

$$\beta = \sigma_{f_1} \cdot \beta' = \sigma_{f_1} \cdot \sigma_{f_2} \cdot \beta'' = \cdots = \sigma_{f_1} \cdot \sigma_{f_2} \cdot \cdots \cdot \sigma_{f_r}.$$

"a positive braid is a sequence of permutations"

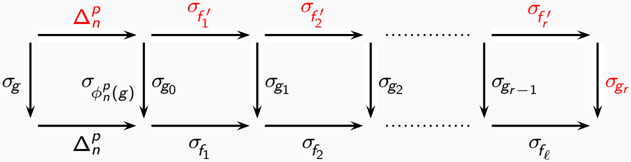
• Fact: *σ_f is a maximal left-divisor of $\sigma_f \cdot \sigma_g$ iff every recoil of f is a descent of g.*

↑ i s.t. $f(i) > f(i{+}1)$ ↑ i s.t. $g^{-1}(i) > g^{-1}(i{+}1)$

• Proposition (Adjan, El-Rifai–Morton, Thurston, ... 1980s): *Every braid in B_n admits a unique expression $\Delta_n^p \sigma_{f_1} \cdots \sigma_{f_r}$ with $p \in \mathbb{Z}$, $f_1 \neq (n, ..., 2, 1)$, $f_r \neq id$, and every recoil of f_{k+1} is a descent of f_k.*

- The point here: not only theoretical, but also tractable.
  - The greedy normal form can be computed efficiently.
  - Key point: computing the normal form of $\sigma_i\beta$ and $\sigma_i^{-1}\beta$ from that of β.

- Recipe:
  - Assume that the normal form of β is $\Delta_n^p\,\sigma_{f_1}\cdots\sigma_{f_r}$; let σ_g be a permutation-braid;



  - The normal form of $\sigma_g\beta$ is $\Delta_n^p\,\sigma_{f'_1}\cdots\sigma_{f'_p}\,\sigma_{g_p}$ if $\sigma_{f'_1}\neq\Delta_n$,
    and $\Delta_n^{p+1}\sigma_{f'_2}\cdots\sigma_{f'_p}\,\sigma_{g_p}$ otherwise.

  - And the normal form of $\sigma_g^{-1}\beta$? There exists g' satisfying $\sigma_g\sigma_{g'} = \Delta_n$,
    hence $\sigma_g^{-1} = \sigma_{g'}\Delta_n^{-1}$, and $\sigma_g^{-1}\beta = \sigma_{g'}\Delta_n^{p-1}\sigma_{f_1}\cdots\sigma_{f_r}$: continue as above.

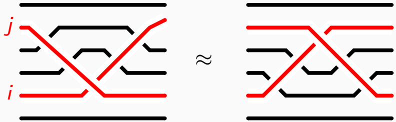
- This corresponds to an automatic structure for B_n (Thurston, Cannon),
  - and, more specifically, to a Garside structure (D.–Paris 1997):

    ↑

    a submonoid B_n^+ of B_n, plus an element Δ_n of B_n^+ such that
    - B_n is a group of fractions for B_n^+,
    - B_n^+ equipped with the (left) divisibility relation is a lattice,
    - $\mathrm{Div}_{\mathrm{left}}(\Delta_n) = \mathrm{Div}_{\mathrm{right}}(\Delta_n)$, $\mathrm{Div}(\Delta_n)$ generates B_n^+, and $\#\mathrm{Div}(\Delta_n) < \infty$.
  - Is the Garside structure on B_n unique? Is there another Garside structure on B_n?

- The dual Garside structure on B_n, based on the Birman–Ko–Lee generators:

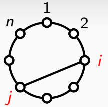
for $1 \leqslant i < j \leqslant n$: $\quad a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$.



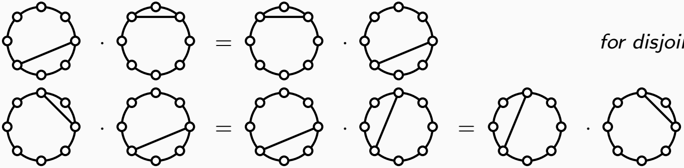
- Definition (Birman–Ko–Lee 1997): B_n^{+*} := submonoid of B_n generated by the $a_{i,j}$s.
  $\Delta_n^* := a_{1,2}a_{2,3} \cdots a_{n-1,n}$ $(= \sigma_1\sigma_2 \cdots \sigma_{n-1})$.

- Proposition: (B_n^{+*}, Δ_n^*) *is a Garside structure on* B_n.
  - a new solution of the Word Problem.

- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$



- Lemma: *In terms of the BKL generators, B_n is presented by the relations*

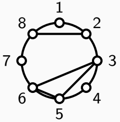


for disjoint chords,

for adjacent chords enumerated in clockwise order.

▶ Hence: For P a p-gon, can define a_P to be the product of the $a_{i,j}$ corresponding to $p-1$ adjacent edges of P in clockwise order;
idem for an union of disjoint polygons.

$a_{2,8}a_{3,5}a_{5,6} \leftrightarrow$



- Proposition (Digne–Michel 2002): *The divisors of Δ_n^* in B_n^{+*} are the $\frac{1}{n+1}\binom{2n}{n}$ elements a_P for P a non-intersecting union of polygons in an n-punctured circle.*

↑
equivalently: a non-crossing partition of $\{1, ..., n\}$

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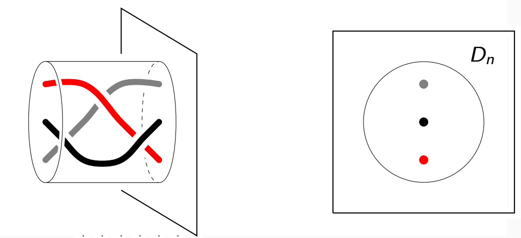
- An n-strand braid diagram = a danse of n points in a disk:

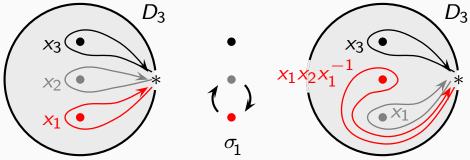


... ▶ an isotopy class of homeomorphisms of D_n leaving ∂D_n fixed

↑ disk with n marked points ↑ boundary of D_n

- Proposition: *The group B_n is (isomorphic to) the mapping class group of D_n.*

- Viewing B_n as a group of (isotopy classes of) homeomorphisms of D_n:
  - action of B_n on the fundamental group of D_n, a free group of rank n.



- From there: a homomorphism ρ from B_n to $\mathrm{Aut}(F_n)$:

$$\rho(\sigma_i) : \begin{cases} x_i & \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} & \mapsto x_i, \\ x_k & \mapsto x_k \quad \text{for } k \neq i, i+1. \end{cases}$$

- Theorem (Artin): *The homomorphism ρ is injective.*
  - a new solution of the Word Problem for B_n (hence of the Braid Isotopy Problem):
    a braid word w represents 1 in B_n iff $\rho(w)(x_k) = x_k$ holds for $k = 1, ..., n$.

- For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and
$$F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1)^+, y_2 - z_1^+, x_2 + y_2^- + (y_1^- + z_1)^-, y_1 + z_1^+),$$
$$F^-(x_1, y_1, x_2, y_2) = (x_1 - y_1^+ - (y_2^+ + z_2)^+, y_2 + z_2^-, x_2 - y_2^- - (y_1^- - z_2)^-, y_1 - z_2^-),$$
with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

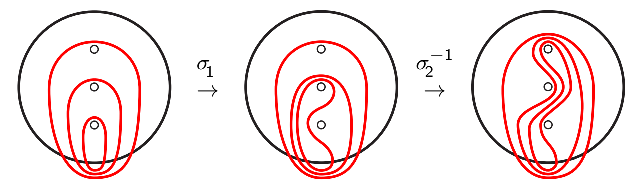
- Define an action of n-strand braid words on $\mathbb{Z}^{2n}$ by
$$(a_1, b_1, \ldots, a_n, b_n) * \sigma_i^e = (a_1', b_1', \ldots, a_n', b_n')$$
with $a_k' = a_k$ and $b_k' = b_k$ for $k \neq i, i{+}1$, and $(a_i', b_i', a_{i+1}', b_{i+1}') = F^e(a_i, b_i, a_{i+1}, b_{i+1})$.

- Definition: *The coordinates of an n-strand braid word w are $(0, 1, 0, 1, \ldots, 0, 1) * w$.*

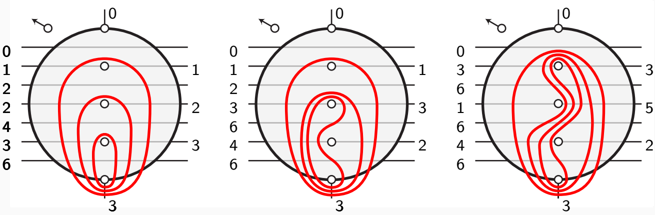
- Theorem (Dynnikov 2000): *The coordinates of w only depend on the braid represented by w, and they characterize the latter.*

  - Hence: a new solution of the Braid Isotopy Problem:
    a braid word w represents 1 iff its Dynnikov coordinates are $(0, 1, 0, 1, \ldots, 0, 1)$.
  - An extremely efficient method: "linear space, quadratic time complexity"

- Braid=homeomorphism of D_n ▸ acts on curves drawn in D_n.



- Count intersections with a fixed triangulation:



▸ $3n+3$ numbers, which determine the braid

- Fact: The Dynnikov coordinates are the half-differences between the previous intersection numbers. (going from $3n + 3$ downto $2n$)

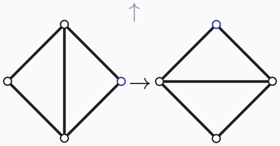
- Problem: Compute the coordinates of $\beta\sigma_i^{\pm 1}$ from those of β and i.
  - compare the intersections of L and $\sigma_i(L)$ with the (fixed) triangulation T

- Main observation:

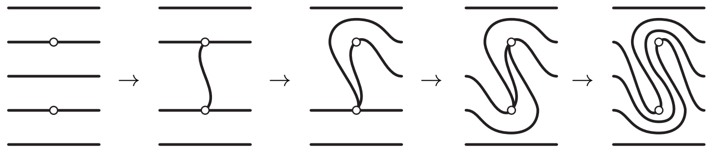
$$\#(\sigma_i(L) \cap T) = \#(L \cap \sigma_i^{-1}(T)).$$

  - compare the intersections of L with T and $\sigma_i^{-1}(T)$.

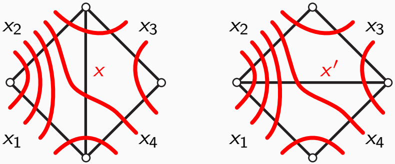
- Lemma: *If T, T' are any two (singular) triangulations, one can go from T to T' using a finite sequence of flips.*



- Hence: One must go from T to $\sigma_i^{-1}(T)$ by a finite sequence of flips.



- For one flip, the formula is



$$x + x' = \max(x_1 + x_3, x_2 + x_4)$$

  - Dynnikov's formulas when iterating four times (four flips).

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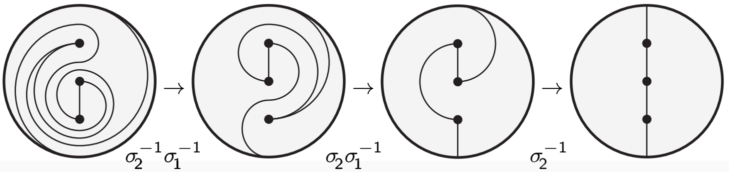
- Here again: n-strand braid = (isotopy class of) homeomorphism of D_n

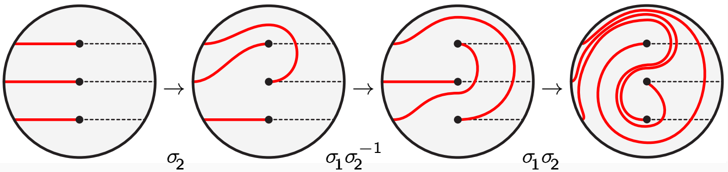
- Principle: Fix one (or several) base curve C,
  - ▶ define a relaxation strategy for unbraiding $\beta(C)$ and coming back to C:
  - ▶ the sequence of $\sigma_i^{\pm 1}$ used to unbraid β gives a distinguished expression of β^{-1} (hence a normal form)
  - ▶ requires to define a complexity notion first.

- Exemple (Fenn et al. 1997, Dynnikov–Wiest 2006):
  C = main diameter of D_n, strategy = consider the "useful arc".



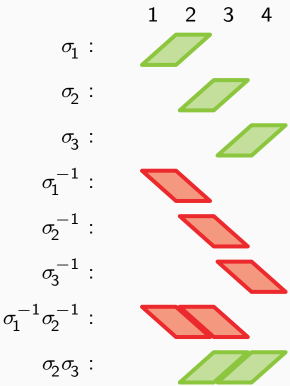
whence $\beta = \sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2$

- Exemple 2 (Bressaud 2005):
  - here C = axes of standard loops
  - strategy: relax $\beta(x_1)$, then $\beta(x_2)$,... by diminishing the number of intersections with half-axes.

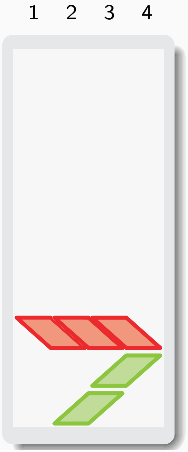


  - a normal form on B_n (whence a solution to the Braid Isotopy Problem),
  - together with an algorithm computing $\mathrm{NF}(w\sigma_i^{\pm 1})$ from $\mathrm{NF}(w)$ and i.

- Remark: The Bressaud normal form has nothing to do with positive braids and B_n^+ (nor with B_n^{+*} either).



etc. (12 pieces:
6 positive, 6 negative)



- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3 \quad = \sigma_2.\sigma_3.\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$.

On the Garside approach:

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On relaxation methods:

• R. Fenn, M.T. Greene, D. Rolfsen, C. Rourke, B. Wiest, *Ordering the braid groups*,
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