Laver tables
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Patrick Dehornoy

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Finite objects with a simple description,
Laver tables

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- Finite objects with a simple description, discovered through set theory,
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Finite objects with a simple description, discovered through set theory, with combinatorial properties that (so far) are only established using unprovable large cardinal hypotheses, and with (potential) applications in low-dimensional topology.
Plan:
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• 1. Combinatorial description of Laver tables
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- 1. Combinatorial description of Laver tables
- 2. Laver tables and set theory
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• 3. Laver tables and low-dimensional topology
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- 1. Combinatorial description of Laver tables
- 2. Laver tables and set theory
- 3. Laver tables and low-dimensional topology
The selfdistributivity law
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- The (left) selfdistributivity law:

\[ x \ast (y \ast z) = (x \ast y) \ast (x \ast z). \]  

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\[ x \star (y \star z) = (x \star y) \star (x \star z). \]  \hspace{1cm} (LD)

cf. associativity: \[ x \star (y \star z) = (x \star y) \star z. \]
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• Classical examples:
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• Classical examples:
  - \( S \) arbitrary and \( x \ast y := y \), or more generally \( x \ast y = f(y) \);
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• Classical examples:

- \( S \) arbitrary and \( x \ast y := y \), or more generally \( x \ast y = f(y) \);
- \( E \) module and \( x \ast y := (1 - \lambda)x + \lambda y \);
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• Remark: These operations obey \( x \ast x = x \) ("idempotency")
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\[ \implies \text{monogenerated substructures are trivial}. \]
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  \( \rightsquigarrow \) monogenerated substructures are trivial.

• Q: Is conjugacy of a free group characterized by selfdistributivity and idempotency?
The selfdistributivity law

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- Remark: These operations obey \( x \ast x = x \) ("idempotency")
  \( \iff \) monogenerated substructures are trivial.

- Q: Is conjugacy of a free group characterized by selfdistributivity and idempotency?
  No (Drápal-Kepka-Musilek 1994, Larue 1999), it obeys
  \[ ((x \ast y) \ast y) \ast (x \ast z) = (x \ast y) \ast ((y \ast x) \ast z), \ldots \]
A Laver table
A Laver table

- A binary operation on \{1, 2, 3, 4\}:
• A binary operation on \{1, 2, 3, 4\}: the four element Laver table
A binary operation on \( \{1, 2, 3, 4\} \): the four element Laver table

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• A binary operation on \(\{1, 2, 3, 4\}\): the four element Laver table

\[
\begin{array}{c|cccc}
  * & 1 & 2 & 3 & 4 \\
 \hline
  1 & 1 & 2 & 3 & 4 \\
  2 & &  &  &  \\
  3 & &  &  &  \\
  4 & &  &  &  \\
\end{array}
\]

• Start with \(1 \mod 4\) in the first column,
• A binary operation on \{1, 2, 3, 4\}: the four element Laver table

\[
\begin{array}{c|cccc}
* & 1 & 2 & 3 & 4 \\
\hline
1 & 2  \\
2 &   \\
3 &   \\
4 &   \\
\end{array}
\]

• Start with \(\text{+1 mod 4}\) in the first column,
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\[
\begin{array}{|c|cccc|}
\hline
\ast & 1 & 2 & 3 & 4 \\
\hline
1 & 2 \\
2 & 3 \\
3 & 4 \\
4 \\
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\end{array}
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\hline
1 & 2 &    &    &    \\
2 & 3 &    &    &    \\
3 & 4 &    &    &    \\
4 & 1 &    &    &    \\
\end{array}
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• Start with \(1 \mod 4\) in the first column,
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A binary operation on \( \{1, 2, 3, 4\} \): the four element Laver table

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\begin{array}{c|cccc}
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\hline
1 & 2 & & & \\
2 & & 3 & & \\
3 & & & 4 & \\
4 & & & & 1 \\
\end{array}
\]

Start with \( +1 \mod 4 \) in the first column, and complete so as to obey the rule \( x \ast (y \ast 1) = (x \ast y) \ast (x \ast 1) \):

\( 4 \ast 2 = \)
A binary operation on \{1, 2, 3, 4\}: the four element Laver table

\[
\begin{array}{c|cccc}
* & 1 & 2 & 3 & 4 \\
\hline
1 & 2 &  &  &  \\
2 & 3 &  &  &  \\
3 & 4 &  &  &  \\
4 & 1 &  &  &  \\
\end{array}
\]

- Start with \( +1 \mod 4 \) in the first column, and complete so as to obey the rule \( x \ast (y \ast 1) = (x \ast y) \ast (x \ast 1) \): \( 4 \ast 2 = 4 \ast (1 \ast 1) \)
A binary operation on \( \{1, 2, 3, 4\} \): the four element \textbf{Laver table}

\[
\begin{array}{c|cccc}
\ast & 1 & 2 & 3 & 4 \\
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1 & 2 & & & \\
2 & & 3 & & \\
3 & & & 4 & \\
4 & & & & 1 \\
\end{array}
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\[
4 \ast 2 = 4 \ast (1 \ast 1) = (4 \ast 1) \ast (4 \ast 1) = 1 \ast 1
\]
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1 & 2 \\
2 & 3 \\
3 & 4 \\
4 & 1 \\
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\[
4 \ast 2 = 4 \ast (1 \ast 1) = (4 \ast 1) \ast (4 \ast 1) = 1 \ast 1 = 2, \]
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\[
4 \ast 2 = 4 \ast (1 \ast 1) = (4 \ast 1) \ast (4 \ast 1) = 1 \ast 1 = 2,
\]

\[
4 \ast 3
\]
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\]

• Start with \(+1 \mod 4\) in the first column, and complete so as to obey the rule \(x * (y * 1) = (x * y) * (x * 1)\):

\[
4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2, \\
4 * 3 = 4 * (2 * 1)
\]
A binary operation on \{1, 2, 3, 4\}: the four element Laver table

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\[
\begin{align*}
4 \star 2 &= 4 \star (1 \star 1) = (4 \star 1) \star (4 \star 1) = 1 \star 1 = 2, \\
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  1 & 2 &  &  &  \\
  2 &  & 3 &  &  \\
  3 &  &  & 4 &  \\
  4 & 1 & 2 &  &  \\
\end{array}
\]

Start with +1\ mod\ 4 in the first column, and complete so as to obey the rule \( x \ast (y \ast 1) = (x \ast y) \ast (x \ast 1) \):

\[
4 \ast 2 = 4 \ast (1 \ast 1) = (4 \ast 1) \ast (4 \ast 1) = 1 \ast 1 = 2,
\]

\[
4 \ast 3 = 4 \ast (2 \ast 1) = (4 \ast 2) \ast (4 \ast 1) = 2 \ast 1
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4 \ast 4 &
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\[
\begin{align*}
4 \ast 2 &= 4 \ast (1 \ast 1) = (4 \ast 1) \ast (4 \ast 1) = 1 \ast 1 = 2, \\
4 \ast 3 &= 4 \ast (2 \ast 1) = (4 \ast 2) \ast (4 \ast 1) = 2 \ast 1 = 3, \\
4 \ast 4 &= 4 \ast (3 \ast 1)
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4 \ast 4 &= 4 \ast (3 \ast 1) = (4 \ast 3) \ast (4 \ast 1)
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4 \ast 4 &= 4 \ast (3 \ast 1) = (4 \ast 3) \ast (4 \ast 1) = 3 \ast 1 = 4,
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\end{array}
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- Start with \(+1 \mod 4\) in the first column, and complete so as to obey the rule \(x \ast (y \ast 1) = (x \ast y) \ast (x \ast 1)\):

\[
\begin{align*}
4 \ast 2 &= 4 \ast (1 \ast 1) = (4 \ast 1) \ast (4 \ast 1) = 1 \ast 1 = 2, \\
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\end{align*}
\]
A binary operation on \( \{1, 2, 3, 4\} \): the four element Laver table

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\[
\begin{array}{c|cccc}
* & 1 & 2 & 3 & 4 \\
\hline
1 & 2 \\
2 & 3 \\
3 & 4 & 4 \\
4 & 1 & 2 & 3 & 4 \\
\end{array}
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\]
• The same construction works for every size
The same construction works for every size and it provides a selfdistributive structure for powers of 2:
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**Proposition (Laver).**— (i) For every $N$, there exists a unique binary operation $*$ on \{1, ..., N\} satisfying

\[
x * 1 = x + 1 \mod N \quad \text{and} \quad x * (y * 1) = (x * y) * (x * 1).
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- **Proposition** (Laver).— (i) For every $N$, there exists a unique binary operation $\ast$ on $\{1, \ldots, N\}$ satisfying

  $x \ast 1 = x + 1 \mod N$ and 
  $x \ast (y \ast 1) = (x \ast y) \ast (x \ast 1)$.

(ii) The operation thus obtained obeys the law

  $x \ast (y \ast z) = (x \ast y) \ast (x \ast z)$ \hspace{1cm} (LD)

if and only if $N$ is a power of 2.
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  (ii) The operation thus obtained obeys the law
  \[
  x * (y * z) = (x * y) * (x * z) \quad \text{(LD)}
  \]
  if and only if $N$ is a power of 2.

$\implies$ the Laver table with 1, 2, 4, 8, 16, 32, ... elements.
Laver tables: examples
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</table>
• For $n \geq 1$, one has $1 \ast 1 = 2 \neq 1$ in $A_n$: not idempotent.
• For $n \geq 1$, one has $1 \ast 1 = 2 \neq 1$ in $A_n$: not idempotent.
  ~~~ quite différent from group conjugacy and other classical LD-structures
• For $n \geq 1$, one has $1 * 1 = 2 \neq 1$ in $A_n$: not idempotent.

\[ \Rightarrow \text{ quite différent from group conjugacy and other classical LD-structures} \]

• **Proposition (Laver).**— The LD-structure $A_n$ is generated by $1$ and admits the presentation $\langle 1 \mid 1_{[2^n]} = 1 \rangle$, with $x_{[k]} = (\ldots((x \ast x) \ast x)\ldots \ast x$, $k$ terms.
• For $n \geqslant 1$, one has $1 \ast 1 = 2 \neq 1$ in $A_n$: not idempotent.
   \[ \Rightarrow \text{quite différent from group conjugacy and other classical LD-structures} \]

• Proposition (Laver).— The LD-structure $A_n$ is generated by $1$ and admits the presentation $\langle 1 \mid 1[2^n] = 1 \rangle$, with $x[k] = (((x \ast x) \ast x) \ast \cdots) \ast x$, $k$ terms.

• Proposition (Drápal).— There exists an (explicit) list of constructions $\mathcal{L}$ (direct product, ...) such that every finite monogenerated LD-structure can be obtained from Laver tables using constructions from $\mathcal{L}$. 
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  ~⇒ quite différent from group conjugacy and other classical LD-structures

- **Proposition (Laver).**— The LD-structure $A_n$ is generated by $1$ and admits the presentation $\langle 1 \mid 1_{[2^n]} = 1 \rangle$, with $x_{[k]} = (\ldots((x \ast x) \ast x)\ldots) \ast x$, $k$ terms.

- **Proposition (Drápal).**— There exists an (explicit) list of constructions $\mathcal{L}$ (direct product, ...) such that every finite monogenerated LD-structure can be obtained from Laver tables using constructions from $\mathcal{L}$.
  ~⇒ think of $\mathbb{Z}/p\mathbb{Z}$ in the associative world
• **Proposition (Laver).**— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2,
• Proposition (Laver).— For every \( p \leq 2^n \), there exists a number \( \pi_n(p) \), a power of 2, such that the \( p \)th row in (the table of) \( A_n \)
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• **Example:**

\[
\begin{array}{cccccccc}
A_3 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 4 & 6 & 8 & 2 & 4 & 6 & 8 \\
2 & 3 & 4 & 7 & 8 & 3 & 4 & 7 & 8 \\
3 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 \\
4 & 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 \\
5 & 6 & 8 & 6 & 8 & 6 & 8 & 6 & 8 \\
6 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 \\
7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p + 1 \mod 2^n$ to $2^n$.

• Example:

<table>
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<tr>
<th>$A_3$</th>
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</table>

$\Rightarrow \pi_3(8) = 8$
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p + 1 \mod 2^n$ to $2^n$. 

<table>
<thead>
<tr>
<th>$A_3$</th>
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</tbody>
</table>

• Example:

$\Rightarrow \pi_3(7) = 1$

$\Rightarrow \pi_3(8) = 8$
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is
the repetition of $\pi_n(p)$ values increasing from $p+1 \mod 2^n$ to $2^n$.

<table>
<thead>
<tr>
<th>$A_3$</th>
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</thead>
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<tr>
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<tr>
<td>2</td>
<td>3 4 7 8 3 4 7 8</td>
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</tr>
<tr>
<td>8</td>
<td>1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>

• Example:

$\Rightarrow \pi_3(6) = 2$
$\Rightarrow \pi_3(7) = 1$
$\Rightarrow \pi_3(8) = 8$
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p + 1 \mod 2^n$ to $2^n$.

**Example:**

<table>
<thead>
<tr>
<th>$A_3$</th>
<th>1 2 3 4 5 6 7 8</th>
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<tbody>
<tr>
<td>1</td>
<td>2 4 6 8 2 4 6 8</td>
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<tr>
<td>2</td>
<td>3 4 7 8 3 4 7 8</td>
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<td>8</td>
<td>1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>

$\Rightarrow \pi_3(5) = 2$  
$\Rightarrow \pi_3(6) = 2$  
$\Rightarrow \pi_3(7) = 1$  
$\Rightarrow \pi_3(8) = 8$
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p + 1 \mod 2^n$ to $2^n$.

<table>
<thead>
<tr>
<th>$A_3$</th>
<th>1 2 3 4 5 6 7 8</th>
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<tbody>
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<td>1</td>
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<td>1 2 3 4 5 6 7 8</td>
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</table>

• Example:

$$\pi_3(4) = 4$$
$$\pi_3(5) = 2$$
$$\pi_3(6) = 2$$
$$\pi_3(7) = 1$$
$$\pi_3(8) = 8$$
- **Proposition (Laver).**— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p + 1 \mod 2^n$ to $2^n$.

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<table>
<thead>
<tr>
<th>$A_3$</th>
<th>1 2 3 4 5 6 7 8</th>
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<tbody>
<tr>
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<td>1 2 3 4 5 6 7 8</td>
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$\Rightarrow \pi_3(3) = 2$

$\Rightarrow \pi_3(4) = 4$

$\Rightarrow \pi_3(5) = 2$

$\Rightarrow \pi_3(6) = 2$

$\Rightarrow \pi_3(7) = 1$

$\Rightarrow \pi_3(8) = 8$
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p+1 \mod 2^n$ to $2^n$.

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$\leadsto \pi_3(2) = 4$  
$\leadsto \pi_3(3) = 2$  
$\leadsto \pi_3(4) = 4$  
$\leadsto \pi_3(5) = 2$  
$\leadsto \pi_3(6) = 2$  
$\leadsto \pi_3(7) = 1$  
$\leadsto \pi_3(8) = 8$
• Proposition (Laver).— For every $p \leq 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the $p$th row in (the table of) $A_n$ is the repetition of $\pi_n(p)$ values increasing from $p + 1 \mod 2^n$ to $2^n$.

• Example:

\[
\begin{array}{|c|cccccccc|}
\hline
\text{A}_3 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 2 & 4 & 6 & 8 & 2 & 4 & 6 & 8 \\
2 & 3 & 4 & 7 & 8 & 3 & 4 & 7 & 8 \\
3 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 \\
4 & 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 \\
5 & 6 & 8 & 6 & 8 & 6 & 8 & 6 & 8 \\
6 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 \\
7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\end{array}
\]

$\Rightarrow \pi_3(1) = 4$

$\Rightarrow \pi_3(2) = 4$

$\Rightarrow \pi_3(3) = 2$

$\Rightarrow \pi_3(4) = 4$

$\Rightarrow \pi_3(5) = 2$

$\Rightarrow \pi_3(6) = 2$

$\Rightarrow \pi_3(7) = 1$

$\Rightarrow \pi_3(8) = 8$
• The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$. 
The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$.

The inverse limit of the $A_n$ is an LD operation on 2-adic numbers;
• The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$.
  
  ~ the inverse limit of the $A_n$ is an LD operation on 2-adic numbers;
  ~ one always has $\pi_n(p) \geq \pi_{n-1}(p)$.
• The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$.
  
  $\leadsto$ the inverse limit of the $A_n$ is an LD operation on 2-adic numbers;
  
  $\leadsto$ one always has $\pi_n(x) \geq \pi_{n-1}(x)$.

• A few values of the periods of 1 and 2:

<table>
<thead>
<tr>
<th>n</th>
<th>$\pi_n(1)$</th>
<th>$\pi_n(2)$</th>
</tr>
</thead>
</table>

Asymptotic behaviour
Asymptotic behaviour

- The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$.
  - the inverse limit of the $A_n$ is an LD operation on 2-adic numbers;
  - one always has $\pi_n(p) \geq \pi_{n-1}(p)$.

- A few values of the periods of 1 and 2:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_n(1)$</td>
<td>1</td>
</tr>
<tr>
<td>$\pi_n(2)$</td>
<td>—</td>
</tr>
</tbody>
</table>
Asymptotic behaviour

• The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$.
  \[ \sim \text{ the inverse limit of the $A_n$ is an LD operation on 2-adic numbers;} \]
  \[ \sim \text{ one always has } \pi_n(p) \geq \pi_{n-1}(p). \]

• A few values of the periods of 1 and 2:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
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<td>$\pi_n(1)$</td>
<td>1</td>
<td>1</td>
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<tr>
<td>$\pi_n(2)$</td>
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<td>2</td>
</tr>
</tbody>
</table>
• The map $x \mapsto x \mod 2^{n-1}$ is a surjective homomorphism from $A_n$ to $A_{n-1}$.

  $\Rightarrow$ the inverse limit of the $A_n$ is an LD operation on 2-adic numbers;

  $\Rightarrow$ one always has $\pi_n(p) \geqslant \pi_{n-1}(p)$.

• A few values of the periods of 1 and 2:

<table>
<thead>
<tr>
<th>$n$</th>
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<th>2</th>
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</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$\pi_n(2)$</td>
<td>$-$</td>
<td>2</td>
<td>2</td>
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  ~ one always has $\pi_n(p) \geq \pi_{n-1}(p)$.

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<th>2</th>
<th>3</th>
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</thead>
<tbody>
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<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$\pi_n(2)$</td>
<td>(-)</td>
<td>2</td>
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• Theorem (Laver, 1995).—
  the answer to the above questions is positive.
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• Theorem (Laver, 1995).— If there exists a selfsimilar set, then the answer to the above questions is positive.
Plan:

1. Combinatorial description of Laver tables
2. Laver tables and set theory
3. Laver tables and low-dimensional topology
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Examples: inaccessible cardinals,
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\begin{align*}
\text{ultra-infinite} & = \text{infinite} \times \text{finite}.
\end{align*}
\]

Examples: inaccessible cardinals, measurable cardinals, etc.

• General principle: “being selfsimilar implies being large”.
  - $A$ is infinite iff $\exists j : A \to A$ injective not bijective;
  
  - $A$ is ultra-infinite (“selfsimilar”) iff $\exists j : A \to A$ injective not bijective and preserving every notion that is definable from $\in$. 

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• Example: \( \mathbb{N} \) infinite, but not ultra-infinite: if \( j : \mathbb{N} \rightarrow \mathbb{N} \) preserves every notion that is definable from \( \in \), then \( j \) preserves 0, 1, 2, etc. hence \( j \) is the identity map.
• **Definition.**— A **rank**
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• **Proposition.**— If $j$ is an embedding of a rank $R$,
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  closure of $\{j\}$ under the “apply” operation: $j(j)$, $j(j)(j)$...
An embedding $j$ maps every ordinal $\alpha$ to an ordinal $j(\alpha) \geq \alpha$ ;
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• An embedding \( j \) maps every ordinal \( \alpha \) to an ordinal \( j(\alpha) \geq \alpha \);
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• Recall: \( j[p] := j(j)(j)\ldots(j) \), \( p \) terms.
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• Recall: \( j_p := j(j)(j)...(j), \ p \) terms.

• Proposition (Laver).— Assume that \( j \) is an embedding of a rank \( R \). For \( k, k' \) in \( \text{Iter}(j) \), declare \( k \equiv_n k' \) if

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Then $\equiv_n$ is a congruence on $\text{Iter}(j)$, it has $2^n$ classes, which are those of $j, j_{[2]}, ..., j_{[2^n]}$. 

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Laver tables: the return

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exact definition of $\equiv_n$ : $\forall x \in R_\gamma (k(x) \cap R_\gamma = k'(x) \cap R_\gamma)$ with $\gamma = \text{crit}(j_{[2^n]})$
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• Hence $\text{Iter}(j)/\equiv_n$ is an LD-structure with $2^n$ elements s.t. $j[p] \ast j = j[p+1 \mod 2^n]$.

• **Corollary.**— The quotient-structure $\text{Iter}(j)/\equiv_n$ is (isomorphic to) the table $A_n$. 
The period of $2$
Lemma 1.— If \( j \) is an embedding, then, for \( m \leq n \) and \( p \leq 2^n \), TFAE
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Lemma 1.— If \( j \) is an embedding, then, for \( m \leq n \) and \( p \leq 2^n \), TFAE
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• **Lemma 2.**— If $j$ is an embedding, then $j(j)(\alpha) \leq j(\alpha)$ holds for every ordinal $\alpha$. 
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• Proof:
The period of 2

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  \[
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  \]  
\( (\ast) \)

Applying \( j \) to \( (\ast) \) gives
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The period of 1

- **Theorem** (Steel, Laver).
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-
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• **Proposition** *(Laver).*— If there exists a selfsimilar set,
• **Theorem (Steel, Laver).**— If $j$ is an embedding of a rank $R$, then the sequence $\text{crit}(j_{[2^n]})$ is unbounded in $R$.

• **Proposition (Laver).**— If there exists a selfsimilar set, the sequence of periods $\pi_n(1)$ tends to $\infty$ with $n$. 
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• **Proposition (Laver).**— If there exists a selfsimilar set, the sequence of periods $\pi_n(1)$ tends to $\infty$ with $n$.

• **Corollary.**— If there exists a selfsimilar set, the substructure generated by $(1, 1, 1, \ldots)$ in the inverse limit of all $\mathbb{A}_n$ is free.
• Did we answer the questions about Laver tables?
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  — No, because the existence of a selfsimilar set is a large cardinal axiom,
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• Is the large cardinal assumption necessary?
  — **Probably not**... So far, we cannot avoid it, but nothing indicates that it should be necessary; and there is no systematic method for avoiding it.

• An attempt: **Drápal**'s program, three steps completed so far...

• A similar example: the orderability of free LD-structures, **first** established using a selfsimilar set, **then** using a direct argument (**based on braid groups**).
Plan:

- 1. Combinatorial description of Laver tables
- 2. Laver tables and set theory
- 3. Laver tables and low-dimensional topology
• Planar diagrams:
• Planar diagrams:
• Planar diagrams:
• Planar diagrams:
Planar diagrams:

\[\rightsquigarrow\text{ projections of curves embedded in } \mathbb{R}^3\]
• Planar diagrams:

\[ \cdots \]

\( \rightsquigarrow \) projections of curves embedded in \( \mathbb{R}^3 \)

• Generic question: recognizing whether two diagrams are (projections of) isotopic figures
• Planar diagrams:

• Generic question: recognizing whether two diagrams are (projections of) isotopic figures

\[ \rightsquigarrow \] find isotopy invariants.
Two diagrams represent isotopic figures \textit{iff} one can go from the former to the latter using finitely many \textbf{Reidemeister moves}:
• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I:
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- type I:

![Reidemeister Move Type I](image-url)
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\[ \sim \quad \_ \quad \_ \quad \sim \]
Reidemeister moves

- Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

  - type I:

  - type II:
• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I:

- type II:
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Reidemeister moves

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- type II:

- type III:
Two diagrams represent isotopic figures \textit{iff} one can go from the former to the latter using finitely many \textbf{Reidemeister moves}:

- type I :

- type II :

- type III :
• Fix a set (of colors) $S$ equipped with two operations $\ast, \bar{\ast}$,
• Fix a set \((\text{of colors})\) \(S\) equipped with two operations \(\ast, \bar{\ast}\), and color the strands in diagrams obeying the rules:

\[
\begin{align*}
  \overset{b}{\rightarrow} & \overset{a}{\rightarrow} \\
  \overset{a}{\rightarrow} & \overset{a \ast b}{\rightarrow}
\end{align*}
\]
• Fix a set (of colors) $S$ equipped with two operations $\ast, \bar{\ast}$, and color the strands in diagrams obeying the rules:

\[
\begin{align*}
&b \rightarrow a \\
&a \rightarrow a \ast b
\end{align*}
\text{ et }
\begin{align*}
&b \rightarrow a \bar{\ast} b \\
&a \rightarrow b
\end{align*}
\]
• Fix a set (of colors) $S$ equipped with two operations $\ast$, $\bar{\ast}$, and color the strands in diagrams obeying the rules:

$$
\begin{align*}
  b & \xrightarrow{\ast} a \\
  a & \xrightarrow{\ast} a \ast b
\end{align*}
$$

et

$$
\begin{align*}
  b & \xrightarrow{\ast} a \bar{\ast} b \\
  a & \xrightarrow{\ast} b
\end{align*}
$$

• Action of Reidemeister moves on colors:
• Fix a set (of colors) $S$ equipped with two operations $\ast, \bar{\ast}$, and color the strands in diagrams obeying the rules:

\[
\begin{align*}
    b & \xrightarrow{a} a \\
    a & \xrightarrow{a \ast b}
\end{align*}
\quad \text{et} \quad \begin{align*}
    b & \xrightarrow{a \bar{\ast} b} \\
    a & \xrightarrow{b}
\end{align*}

• Action of Reidemeister moves on colors:

\[
\begin{align*}
    c & \xrightarrow{b} \\
    b & \xrightarrow{a}
\end{align*}
\]
• Fix a set (of colors) $S$ equipped with two operations $\ast$, $\bar{\ast}$, and color the strands in diagrams obeying the rules:

\[ \begin{array}{ccc}
    b & \longrightarrow & a \\
    a & \longrightarrow & a \ast b \\
    b & \longrightarrow & a \bar{\ast} b
\end{array} \]

et

\[ \begin{array}{ccc}
    b & \longrightarrow & a \\
    a & \longrightarrow & b
\end{array} \]

• Action of Reidemeister moves on colors:

\[ \begin{array}{ccc}
    c & \longrightarrow & b \\
    b & \longrightarrow & b \ast c \\
    a & \longrightarrow & a \ast (b \ast c)
\end{array} \]
• Fix a set (of colors) $S$ equipped with two operations $\ast, \bar{\ast}$, and color the strands in diagrams obeying the rules:

\[
\begin{align*}
    b & \longrightarrow a & \text{et} & b & \longrightarrow a \ast b \\
    a & \longrightarrow a \ast b & & a & \longrightarrow b
\end{align*}
\]

• Action of Reidemeister moves on colors:

\[
\begin{align*}
    c & \longrightarrow b & \longrightarrow a & \text{et} & c & \longrightarrow b \longrightarrow a \\
    b & \longrightarrow b \ast c & \longrightarrow a \ast b & \sim & b & \longrightarrow a \\
    a & \longrightarrow a \ast (b \ast c)
\end{align*}
\]
• Fix a set (of colors) $S$ equipped with two operations $\ast, \overline{\ast}$, and color the strands in diagrams obeying the rules:

\[
\begin{align*}
    b & \ast a \\
    a & \ast b \\
\end{align*}
\hspace{1cm}
\begin{align*}
    b & \overline{\ast} a \\
    a & \overline{\ast} b \\
\end{align*}
\]

• Action of Reidemeister moves on colors:

\[
\begin{align*}
    c & \ast b \\
    b & \ast c \\
    a & \ast (b \ast c) \\
\end{align*}
\hspace{1cm}
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• Fix a set (of colors) $S$ equipped with two operations $\ast, \bar{\ast}$, and color the strands in diagrams obeying the rules:

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\end{align*}
$$

et

$$
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&a \rightarrow b
\end{align*}
$$

• Action of Reidemeister moves on colors:

$$
\begin{align*}
&c \rightarrow b \rightarrow a \\
&b \rightarrow b \ast c \rightarrow a \ast b
\end{align*}
$$

$\sim$

$$
\begin{align*}
&c \rightarrow a \rightarrow a \ast c \\
&a \rightarrow a \ast b
\end{align*}
$$

$\sim$

Hence: $S$-colorings invariant under Reidemeister move III $\Leftrightarrow (S, \ast)$ LD-structure
• Idem for Reidemeister move II:
Idem for Reidemeister move II:

There exists $\bar{*}$ satisfying $x \ast (x \ast y) = y$ and $x \ast (x \ast y) = y$
iff the left-translations of $(S, \ast)$ are bijections.
• Idem for Reidemeister move II:

\[
\begin{array}{c}
\text{a} \\
\text{a} \ast \text{b} \\
\text{a} \\
\text{a} \ast (\text{a} \ast \text{b}) \\
\text{b} \\
\end{array}
\]

There exists \( \ast \) satisfying \( x \ast (x \ast y) = y \) and \( x \ast (x \ast y) = y \)
iff the left-translations of \((S, \ast)\) are bijections.

\(\Rightarrow\) Hence: \( S \)-colorings invariant under Reidemeister moves II+III \(\Leftrightarrow\)
\[(S, \ast)\] is an LD-structure with bijective left-translations
• Idem for Reidemeister move II:

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$\Rightarrow$ Hence: $S$-colorings invariant under Reidemeister moves II+III $\iff$
$(S, \ast)$ is an LD-structure with bijective left-translations

a rack (Fenn–Rourke)
- Idem for Reidemeister move II:

\[
\begin{align*}
  b & \leadsto a \leadsto a \ast (a \ast b) \\
  a & \leadsto a \ast b \leadsto a
\end{align*}
\]

There exists \( \ast \) satisfying \( x \ast (x \ast y) = y \) and \( x \ast (x \ast y) = y \) iff the left-translations of \((S, \ast)\) are bijections.

\[\Rightarrow\] Hence: \( S \)-colorings invariant under Reidemeister moves II+III \( \iff \) \((S, \ast)\) is an LD-structure with bijective left-translations

\[
\text{a rack (Fenn–Rourke)}
\]

- Idem for Reidemeister move I:

\[
\begin{align*}
  a & \leadsto a \ast a \leadsto a \\
  a & \leadsto a \ast a \leadsto a \ast a
\end{align*}
\]
• Idem for Reidemeister move II:

\[
\begin{align*}
\text{b} &\xrightarrow{a} \text{a} \xrightarrow{a \ast (a \ast b)} \text{b} \\
\text{a} &\xrightarrow{a \ast b} \text{a} \\
\end{align*}
\]

There exists \( \bar{\ast} \) satisfying \( x \ast (x \ast y) = y \) and \( x \ast (x \ast y) = y \)
iff the left-translations of \((S, \ast)\) are bijections.

\[\Rightarrow \text{ Hence: } S\text{-colorings invariant under Reidemeister moves II+III } \iff (S, \ast) \text{ is an LD-structure with bijective left-translations} \]
\[\text{a rack (Fenn–Rourke)}\]

• Idem for Reidemeister move I:

\[
\begin{align*}
\text{a} &\xrightarrow{a} \text{a} \ast \text{a} \\
\text{a} &\xrightarrow{a} \text{a} \xrightarrow{a} \text{a} \xrightarrow{\bar{a} \ast \text{a}} \text{a} \\
\end{align*}
\]

\[\Rightarrow \text{ Hence: } S\text{-colorings invariant under Reidemeister moves I+II+III } \iff (S, \ast) \text{ is an idempotent rack}\]
• Idem for Reidemeister move II:

There exists $\bar{\ast}$ satisfying $x \ast (x \bar{\ast} y) = y$ and $x \bar{\ast} (x \ast y) = y$
iff the left-translations of $(S, \ast)$ are bijections.

Hence: $S$-colorings invariant under Reidemeister moves II+III $\iff$
$(S, \ast)$ is an LD-structure with bijective left-translations
\[ \uparrow \]
a rack (Fenn–Rourke)

• Idem for Reidemeister move I:

Hence: $S$-colorings invariant under Reidemeister moves I+II+III $\iff$
$(S, \ast)$ is an idempotent rack
\[ \uparrow \]
a quandle (Joyce)
• Theoretical (Joyce, Matveev): The “fundamental quandle” is a complete invariant w.r.t. isotopy up to mirror symmetry.
Cocycles

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Theoretical (Joyce, Matveev): The “fundamental quandle” is a complete invariant w.r.t. isotopy up to mirror symmetry.

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Definition. A 2-cocycle on an LD-structure $(S, \ast)$ is a map $\phi : S^2 \to \mathbb{Z}$ satisfying $\phi(x, z) + \phi(x \ast y, x \ast z) = \phi(y, z) + \phi(x, y \ast z)$. 
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• **Definition.** — A 2-cocycle on an LD-structure \((S, \ast)\) is a map \(\phi : S^2 \to \mathbb{Z}\) satisfying \(\phi(x, z) + \phi(x \ast y, x \ast z) = \phi(y, z) + \phi(x, y \ast z)\).

• Every 2-cocycle provides an invariant w.r.t. Reidemeister move III (and more...):
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• **Definition.**— A 2-cocycle on an LD-structure $(S, *)$ is a map $\phi : S^2 \to \mathbb{Z}$ satisfying $\phi(x, z) + \phi(x*y, x*z) = \phi(y, z) + \phi(x, y*z)$.

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Cocycles

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![Diagram](attachment:image.png)
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- Every 2-cocycle provides an invariant w.r.t. Reidemeister move III (and more...):
• Laver tables are LD-structures, but neither racks (nor quandles):
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  not obvious to use them in topology,
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Proposition (D., Lebed).— The 2-cocycles for $A_n$ make a free $\mathbb{Z}$-module of rank $2^n$, 
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<td>1 1</td>
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</tr>
<tr>
<td>5</td>
<td>1</td>
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<tr>
<td>6</td>
<td>1</td>
<td>1 1</td>
<td>1</td>
<td>1 1</td>
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<tr>
<td>7</td>
<td>1</td>
<td>1 1</td>
<td>1</td>
<td>1 1</td>
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<td>1 1</td>
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<td>8</td>
<td>1</td>
<td>1 1</td>
<td>1</td>
<td>1 1</td>
<td>1</td>
<td>1 1</td>
<td>1</td>
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</tbody>
</table>
Laver tables are LD-structures, but neither racks (nor quandles):

- not obvious to use them in topology, but possible (Przytycki, ...),
- step 1 : determine the associated cocycles.

**Proposition (D., Lebed).**— The 2-cocycles for $\mathcal{A}_n$ make a free $\mathbb{Z}$-module of rank $2^n$, with an explicit basis made of $\{0, 1\}$-valued functions.

\[
\begin{align*}
\psi_{1,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad 1 \ldots \ldots \\
5 & \quad 1 \ldots \ldots \\
6 & \quad 1 \ldots \ldots \\
7 & \quad 1 \ldots \ldots \\
8 & \quad \ldots \ldots \\
\psi_{2,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad 1 \ldots \ldots \\
5 & \quad 1 \ldots \ldots \\
6 & \quad 1 \ldots \ldots \\
7 & \quad 1 \ldots \ldots \\
8 & \quad \ldots \ldots \\
\psi_{3,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad 1 \ldots \ldots \\
5 & \quad 1 \ldots \ldots \\
6 & \quad 1 \ldots \ldots \\
7 & \quad 1 \ldots \ldots \\
8 & \quad \ldots \ldots \\
\psi_{4,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad 1 \ldots \ldots \\
5 & \quad 1 \ldots \ldots \\
6 & \quad 1 \ldots \ldots \\
7 & \quad 1 \ldots \ldots \\
8 & \quad \ldots \ldots \\
\psi_{5,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad \ldots \ldots \\
5 & \quad 1 \ldots \ldots \\
6 & \quad 1 \ldots \ldots \\
7 & \quad 1 \ldots \ldots \\
8 & \quad \ldots \ldots \\
\psi_{6,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad \ldots \ldots \\
5 & \quad 1 \ldots \ldots \\
6 & \quad 1 \ldots \ldots \\
7 & \quad 1 \ldots \ldots \\
8 & \quad \ldots \ldots \\
\psi_{7,3} & \quad 12345678 \\
1 & \quad 1 \ldots \ldots \\
2 & \quad 1 \ldots \ldots \\
3 & \quad 1 \ldots \ldots \\
4 & \quad \ldots \ldots \\
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\end{align*}
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s.t. \( \psi_n(x, y) = 1 \) iff \( y \) is a multiple of the period of \( x \) in \( A_n \).
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$$\exists z \left( y = z \ast x \right)$$

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![Diagram](image.png)

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• **Question**: What do these new positive braid invariants count?
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• **Question**: What do these new positive braid invariants count?

• **Conclusion**: Reasonable hope of applying Laver tables in low-dimensional topology.
• Are the properties of periods in Laver tables an application of set theory?
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Richard Laver
(1942-2012)
• R. Laver, *On the algebra of elementary embeddings of a rank into itself,*

• P. Dehornoy, *Braids and self-distributivity,*
  Progress in math. vol 192, Birkhaüser (1999), chapters X and XIII

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