Braid combinatorics, permutations, and noncrossing partitions

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- A few combinatorial questions involving braids and their Garside structures:
  - the classical Garside structure, connected with permutations,
  - the dual Garside structure, connected with noncrossing partitions.
• Plan:
  1. Braid combinatorics: Artin generators
  2. Braid combinatorics: Garside generators
Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
- a **4-strand braid diagram**

  = 2D-projection of a 3D-figure:

  ![Diagram](image)

- **isotopy** = move the strands but keep the ends fixed:

  ![Isotopy Example](image)

- a **braid** := an isotopy class

  represented by 2D-diagram,

  but different 2D-diagrams may give rise to the same braid.
• **Product** of two braids:

\[ \begin{array}{ccc}
\text{braid} & * & \text{braid} \\
\cline{1-3}
\text{braid} & := & \text{braid}
\end{array} \]

• Then well-defined (with respect to isotopy), associative, admits a unit:

\[ \begin{array}{ccc}
\text{braid} & * & \text{braid} \\
\cline{1-3}
\text{braid} & = & \text{braid}
\end{array} \] \simeq \text{isotopic to}

and inverses:

\[ \begin{array}{ccc}
\text{braid} & * & \text{braid} \\
\cline{1-3}
\text{braid} & = & \text{braid}
\end{array} \] \simeq \text{isotopic to}

► For each \( n \), the group \( B_n \) of \( n \)-strand braids (E. Artin, 1925).
• Artin generators of $B_n$:

\[
\sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1}
\]

• **Theorem (Artin).**— The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

\[
\begin{align*}
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| \geq 2.
\end{align*}
\]
• For $n \geq 2$, the group $B_n$ is infinite $\Rightarrow$ consider finite subsets.

• $B_n^+ :=$ monoid of classes of $n$-strand positive diagrams
  all crossings have a positive orientation

• **Theorem (Garside, 1967).**— As a monoid, $B_n^+$ admits the presentation... (as $B_n$);
  it is cancellative, and admits lcm$s$ and gcd$s$.

• Hence: Equivalent positive braid words have the same length,
  $\Rightarrow$ every positive braid $\beta$ has a well-defined length $\|\beta\|^\text{Art}$ w.r.t. Artin generators $\sigma_i$.

• **Question:** Determine $N_{n,\ell}^{\text{Art}^+} := \# \{ \beta \in B_n^+ \mid \|\beta\|^\text{Art} = \ell \}$
  and/or the associated generating series.
• **Theorem (Deligne, 1972).**— For every \( n \), the g.f. of \( N_{n,\ell}^{\text{Art}+} \) is rational.

Proof: For \( \beta \) in \( B_n^+ \), define \( M(\beta) := \{ \beta \gamma \mid \gamma \in B_n^+ \} = \text{right-multiples of } \beta \).

- Then \( B_n^+ \setminus \{1\} = \bigcup_i M(\sigma_i) \), and \( M(\sigma_i) \cap M(\sigma_j) = M(\text{lcm}(\sigma_i, \sigma_j)) \).
- By inclusion–exclusion, get induction \( N_{n,\ell}^{\text{Art}+} = c_1 N_{n,\ell-1}^{\text{Art}+} + \cdots + c_K N_{n,\ell-K}^{\text{Art}+} \). \( \square \)

More precisely: for every \( n \), the generating series of \( N_{n,\ell}^{\text{Art}+} \) is the inverse of a polynomial \( P_n(t) \).

• **Proposition (Bronfman, 2001).**— Starting from \( P_0(t) = P_1(t) = 1 \), one has
\[
P_n(t) = \sum_{i=1}^n (-1)^{i+1} t^{\frac{i(i-1)}{2}} P_{n-i}(t).
\]
• Same question for $B_n$ instead of $B_n^+$; all representatives don’t have the same length
  ▶ define $\|\beta\|^{\text{Art}} :=$ the **minimal** length of a word representing $\beta$.

• **Question:** Determine $N_{n,\ell}^{\text{Art}} := \#\{\beta \in B_n \mid \|\beta\|^{\text{Art}} = \ell\}$
  and/or determine the associated generating series.

• **Proposition** (Mairesse–Matheus, 2005).— The generating series of $N_{3,\ell}^{\text{Art}}$ is
  \[
  1 + \frac{2t(2-2t-t^2)}{(1-t)(1-2t)(1-t-t^2)}.\]

• Then open, even $N_{4,\ell}^{\text{Art}}$ : (Mairesse) no rational fraction with degree $\leq 13$ denominator.

• “Explanation”: Artin generators are not the right generators…
  ▶ change generators
Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
• **Definition:** A *Garside structure* in a group $G$ is a subset $S$ of $G$ s.t. every element $g$ of $G$ admits an *$S$-normal* decomposition, meaning $g = s_p^{-1} \cdots s_1^{-1} t_1 \cdots t_q$ with $s_1, \ldots, s_p, t_1, \ldots, t_q$ in $S$ and, using “$f$ left-divides $g$” for “$f^{-1}g$ lies in the submonoid $\hat{S}$ of $G$ generated by $S$”,
  ▶ every element of $S$ left-dividing $s_i s_{i+1}$ left-divides $s_i$,
  ▶ every element of $S$ left-dividing $t_i t_{i+1}$ left-divides $t_i$,
  ▶ 1 is the only element of $S$ left-dividing $s_1$ and $t_1$.

• When it exists, an $S$-normal decomposition is *(essentially)* unique, and geodesic.

• Every group is a Garside structure in itself: interesting only when $S$ is small.

• Normality is *local*: if $S$ is finite, $S$-normal sequences make a rational language
  ▶ automatic structure, solution of the word and conjugacy problems, ...
  ▶ counting problems: $\#$ elements with $S$-normal decompositions of length $\ell$.

• **Definition:** A Garside structure $S$ in a group $G$ is *bounded* if there exists an element $\Delta$ (*“Garside element”*) such that $S$ consists of the left-divisors of $\Delta$ in $\hat{S}$.

• In this case:
  ▶ the $S$-normal decomposition of $g$ in $\hat{S}$ is recursively given by $s_1 = \gcd(g, \Delta)$;
  ▶ $(s, t)$ is $S$-normal iff 1 is the only element of $S$ left-dividing $s^{-1} \Delta$ and $t$. 
• **Permutation** associated with a braid:

\[
\begin{array}{c}
3 & 4 \\
1 & 3 \\
2 & 2 \\
4 & 1
\end{array}
\quad \mapsto \quad (4, 2, 1, 3)
\]

- A surjective homomorphism \( \pi_n : B_n \rightarrow S_n \).

• **Lemma**: Call a braid **simple** if it can be represented by a positive diagram in which any two strands cross at most once. Then, for every permutation \( f \) in \( S_n \), there exists exactly one simple braid \( \sigma_f \) satisfying \( \pi_n(\sigma_f) = f \).

\[
(4, 2, 1, 3) \quad \mapsto \quad (4, 2, 1, 3) \quad : \quad \sigma_1 \sigma_2 \sigma_1 \sigma_3
\]

- The family \( S_n \) of all simple \( n \)-strand braids is a copy of \( S_n \).
• **Theorem** (Garside, Adjan, Morton–ElRifai, Thurston).— For each $n$, the family $S_n$ is a Garside structure in $B_n$, bounded by $\sigma_{(n,\ldots,1)}$; the associated monoid is $B_n^+$.

• “Garside’s fundamental braid” $\Delta_n := \sigma_{(n,\ldots,1)}$, whence $\Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1$:
  
  $\Delta_1 = 1, \; \Delta_2 = \sigma_1, \; \Delta_2 = \sigma_1 \sigma_2 \sigma_1, \; \text{ etc.}$

• A new family of generators: the **Garside** generators $\sigma_f$
  
  ▶ a very redundant family: $n!$ elements, whereas only $n-1$ Artin generators;
  
  ▶ many expressions for a braid, but a distinguished one: the $S_n$-normal one;
  
  ▶ in terms of Garside generators, the group $B_n$ – and the monoid $B_n^+$ – are presented by the relations $\sigma_f \sigma_g = \sigma_{fg}$ with $\ell(f) + \ell(g) = \ell(fg)$;

  - length of $f := \#$ of inversions in $f$

  ▶ the poset $(S_n, \preceq)$ is isomorphic to $(\mathfrak{S}_n, \leq)$.

  - left-divisibility in $B_n^+$
  
  - weak order in $\mathfrak{S}_n$
• **Question:** Determine $N^{\text{Gar}^+}_{n, \ell} := \# \{ \beta \in B^+_n \mid \|\beta\|^{\text{Gar}} = \ell \}$ and/or its generating series, where $\|\beta\|^{\text{Gar}} := \text{length of the } S_n\text{-normal decomposition.}$

(and idem with $N^{\text{Gar}_n}_{n, \ell} := \# \{ \beta \in B_n \mid \|\beta\|^{\text{Gar}} = \ell \}$.)

• An easy question (contrary to the case of Artin generators):
  ▶ by construction, $N^{\text{Gar}^+}_{n, \ell} = \# \text{ length } \ell \text{ normal sequences in } B^+_n,$
  ▶ and normality is a local property:
    a sequence is $S_n\text{-normal iff every length } 2 \text{ subsequence is } S_n\text{-normal.}$

**Proposition.**— Let $M_n$ be the $n! \times n!$ matrix indexed by simple braids (i.e., by permutations) s.t. $(M_n)_{s, t} = \begin{cases} 1 & \text{ if } (s, t) \text{ is normal,} \\ 0 & \text{ otherwise.} \end{cases}$

Then $N^{\text{Gar}^+}_{n, \ell}$ is the idth entry in $(1, \ldots, 1) \cdot M_n^\ell$.

▶ For each $n$, the generating series of $N^{\text{Gar}^+}_{n, \ell}$ is rational.
Reducing the size of the matrix

- **Lemma 1**: For \( f, g \) in \( S_n \), the pair \((\sigma_f, \sigma_g)\) is normal iff \( \text{Desc}(f) \supseteq \text{Desc}(g^{-1}) \).
  \[
  \text{descents of } f := \{k | f(k) > f(k+1)\}
  \]

- Hence, if \( \text{Desc}(g^{-1}) = \text{Desc}(g'^{-1}) \), the columns of \( g \) and \( g' \) in \( M_n \) are equal;
  - columns can be gathered: replace \( M_n \) (size \( n! \)) with \( M'_n \) (size \( 2^{n-1} \)).

- **Lemma 2**: The \# of permutations \( f \) satisfying \( \text{Desc}(f) \supseteq I \) and \( \text{Desc}(f^{-1}) \supseteq J \) is the \# of \( k \times \ell \) matrices with entries in \( \mathbb{N} \) s.t. the sum of the \( i \)th row is \( p_i \) and the sum of the \( j \)th column is \( q_j \), with \((p_1, ..., p_k)\) the composition of \( I \) and \((q_1, ..., q_\ell)\) that of \( J \).

- Hence \((M'_n)_{I,J}\) only depends on the partition of \( J \);
  - can gather columns again: replace \( M'_n \) (size \( 2^{n-1} \)) with \( M''_n \) (size \( p(n) \)).

- Remarks:
  - Going from \( M_n \) to \( M''_n \approx \) reducing the size of the automatic structure of \( B_n \) from \( n! \) to \( p(n) \) (\( \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2n/3}} \))
  - (Hohlweg) That \((M'_n)_{I,J}\) only depends on the partition of \( J \) is (another) form of Solomon's result about the descent algebra.
• The growth rate of $N_{n,\ell}^\text{Gar}^+$ is connected with the eigenvalues of $M_n$, hence of $M_n''$:

CharPol($M_1''$) = $x - 1$
CharPol($M_2''$) = CharPol($M_1''$) · ($x - 1$)
CharPol($M_3''$) = CharPol($M_2''$) · ($x - 2$)
CharPol($M_4''$) = CharPol($M_3''$) · ($x^2 - 6x + 3$)
CharPol($M_5''$) = CharPol($M_4''$) · ($x^2 - 20x + 24$),...

• **Theorem (Hivert–Novelli–Thibon).**—

The characteristic polynomial of $M_n''$ divides that of $M_{n+1}''$.

▶ Proof: Interpret $M_n''$ in terms of quasi-symmetric functions in the sense of Malvenuto–Reutenauer, and determine the LU-decomposition. □

• Spectral radius:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(M_n)$</td>
<td>1</td>
<td>2</td>
<td>5.5</td>
<td>18.7</td>
<td>77.4</td>
<td>373.9</td>
<td>2066.6</td>
</tr>
<tr>
<td>$\rho(M_n)/(n\rho(M_{n-1}))$</td>
<td>0.5</td>
<td>0.667</td>
<td>0.681</td>
<td>0.687</td>
<td>0.689</td>
<td>0.690</td>
<td>0.691</td>
</tr>
</tbody>
</table>

▶ What is the asymptotic behaviour?
• So far: $N_{n,\ell}^{\text{Gar}^+}$ with $n$ fixed and $\ell$ varying; for $\ell$ fixed and $n$ varying, different induction schemes (starting with $N_{n,1}^{\text{Gar}^+} = n!$).

• Proposition. — $N_{n,2}^{\text{Gar}^+} = \sum_{0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 N_{i,2}^{\text{Gar}^+}$, hence (Carlitz–Scoville–Vaughan) $1 + \sum_{n} N_{n,2}^{\text{Gar}^+} \frac{z^n}{(n!)^2} = \frac{1}{J_0(\sqrt{z})}$. Bessel function $J_0$

• Put $N_{n,\ell}^{\text{Gar}^+}(s) := \#$ normal sequences in $B_n^+$ finishing with $s$:
  $N_{n,3}^{\text{Gar}^+}(\Delta_{n-1}) = 2^{n-1}$, $N_{n,3}^{\text{Gar}^+}(\Delta_{n-2}) \sim 2 \cdot 3^n$, $N_{n,4}^{\text{Gar}^+}(\Delta_{n-1}) = \lfloor n!e \rfloor - 1$

• Conclusion: Braid combinatorics w.r.t. Garside generators leads to new, interesting (?) questions about permutation combinatorics.
• Braid groups are countable, braids can be encoded in integers, and most of their (algebraic) properties can be proved in the logical framework of Peano arithmetic, and even of weaker subsystems, like $I\Sigma_1$ where induction is limited to formulas involving at most one unbounded quantifier.

• Braids admit an ordering, s.t. $(B^+_n, \leq)$ is a well-ordering of type $\omega \cdot \omega^{n-2}$;
  ▶ one can construct long (finite) descending sequences of positive braids;
  ▶ but this cannot be done in $I\Sigma_1$ (reminiscent of Goodstein’s sequences);
  ▶ where is the transition from $I\Sigma_1$-provability to $I\Sigma_1$-unprovability?

• Definition: For $F : \mathbb{N} \to \mathbb{N}$, let $WO_F$ be the statement:
  "For every $\ell$, there exists $m$ s.t. every strictly decreasing sequence $(\beta_t)_{t \geq 0}$ in $B^+_3$ satisfying $\|\beta_t\|_{Gar} \leq \ell + F(t)$ for each $t$ has length at most $m$".

• $WO_0$ trivially true (finite #), and $WO_F$ provable for every $F$ using König’s Lemma.

• Theorem (Carlucci, D., Weiermann).— For $r \leq \omega$, let $F_r(x) := \lceil Ack^{-1}(x) \sqrt{x} \rceil$. Then $WO_{F_r}$ is $I\Sigma_1$-provable for finite $r$, and $I\Sigma_1$-unprovable for $r = \omega$.

  ▶ Proof: Evaluate $\#\{\beta \in B^+_3 \mid \|\beta\|_{Gar} \leq \ell \& \beta < \Delta^k_3\}$. □
Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
• Another family of generators for $B_n$: the Birman–Ko–Lee generators

$$a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_{i} \sigma_{i+1} \cdots \sigma_{j-1}$$

for $1 \leq i < j \leq n$.

• The dual braid monoid: the submonoid $B^{++}_n$ of $B_n$ generated by the elements $a_{i,j}$.

• Proposition (Birman–Ko–Lee, 1997).— Let $\delta_n = \sigma_{n-1} \cdots \sigma_2 \sigma_1$. Then the family of all divisors of $\delta_n$ in $B^{++}_n$ is a Garside structure in $B_n$; it is bounded by $\delta_n$. 
Chord representation of the Birman–Ko–Lee generators:

\[
\begin{align*}
\text{Lemma:} & \quad \text{In terms of the BKL generators, } B_n \text{ is presented by the relations} \\
& \quad \text{for disjoint chords,} \\
& \quad \text{for adjacent chords enumerated in clockwise order.}
\end{align*}
\]

Hence: For \( P \) a \( p \)-gon, can define \( a_P \) to be the product of the \( a_{i,j} \) corresponding to \( p-1 \) adjacent edges of \( P \) in clockwise order; \textit{idem} for an union of disjoint polygons.
**Proposition (Bessis–Digne–Michel).—** The elements of the Garside structure $S_n^*$ (divisors of $\delta_n$ in $B_n^{++*}$) are the elements $a_P$ with $P$ a union of disjoint polygons with $n$ vertices, hence in 1-1 correspondence with the $\text{Cat}_n$ noncrossing partitions of $\{1, \ldots, n\}$.

- notation $a_\lambda$ for $\lambda$ a noncrossing partition

**Examples:**

- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\} \leftrightarrow a_{2,8}a_{3,5}a_{5,6}$

- $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\} \leftrightarrow \delta_8 = a_{12}a_{23} \cdots a_{78}$

**Remark:** The permutation of the braid $a_\lambda$ is the permutation associated with $\lambda$ (product of cycles of the parts)
• **Question:** Determine $N_{n,\ell}^{\text{BKL}^+} := \#\{\beta \in B_n^+ \mid \|\beta\|^{\text{BKL}} = \ell\}$ and its generating series, where $\|\beta\|^{\text{BKL}} :=$ length of the $S_n^*$-normal decomposition of $\beta$.

• For instance: $N_{n,1}^{\text{BKL}^+} = \# S_n^* = \text{Cat}_n$.

• Exactly similar to the classical case: *local* property, etc.

---

**Proposition.**— Let $M_n^*$ be the $\text{Cat}_n \times \text{Cat}_n$ matrix indexed by noncrossing partitions s.t. $(M_n^*)_{\lambda,\mu} = \begin{cases} 1 & \text{if } (a_\lambda, a_\mu) \text{ is } S_n^*\text{-normal,} \\ 0 & \text{otherwise.} \end{cases}$ Then $N_{n,\ell}^{\text{BKL}^+}$ is the $1_n$th entry in $(1, \ldots, 1) \cdot M_n^* \ell$. For every $n$, the generating series of $N_{n,\ell}^{\text{BKL}^+}$ is rational.
• When is \((a_\lambda, a_\mu) S_n^\ast\)-normal?

• Recall: If a Garside structure \(S\) is bounded by \(\Delta\), then \((s, t)\) is \(S\)-normal iff \(\partial s\) and \(t\) have no nontrivial common left-divisor.
  ▶ When does \(a_{i,j}\) left-divide \(a_\lambda\)?
  ▶ What is the partition of \(\partial a_\lambda\) in terms of that of \(a_\lambda\)?

• **Lemma (Bessis–Digne–Michel):** The element \(a_{i,j}\) left- (or right-) divides \(a_\lambda\) iff the chord \((i, j)\) is included in the polygon of \(\lambda\).

• **Lemma (Bessis–Digne–Michel):** The partition of \(\partial a_\lambda\) is the Kreweras complement \(\overline{\lambda}\) of \(\lambda\).
• **Proposition** (Biane).— The generating series $G(z)$ of $N^\text{BKL}_n^{+}$ is derived from the generating series $F(z)$ of $\text{Cat}_n^2$ by

$$G(z) = F(zG(z)). \quad (\#)$$

• Proof:

  ▶ Let $G(z) = \sum_n N^\text{BKL}_n^{+} z^n$

  with $N^\text{BKL}_n^{+} = \#$ length 2 normal sequences $= \#$ positive entries in $M_n^\times$.

  ▶ From what we saw: $(M_n^\times)_{\lambda, \mu} = 1$ iff $\overline{\lambda} \land \mu = 0_n$. As $\lambda \rightarrow \overline{\lambda}$ is a bijection, one has also $N^\text{BKL}_n^{+} = \#\{(\lambda, \mu) \in (\text{NC}_n)^2 \mid \lambda \lor \mu = 1_n\}$.

  ▶ The number $N^\text{BKL}_n^{+}$ is the $n$th free cumulant of $X_1^2 X_2^2$

  where $X_1, X_2$ are independent free random variables of variance 1.

  ▶ Hence connected to the g.f. $F$ of pairs of noncrossing partitions under $(\#)$. □
### First values:

<table>
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<tr>
<td>$\mathcal{N}^{BKL+}_{2,d}$</td>
<td>2</td>
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<td>4</td>
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<td>1 515</td>
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<td>99</td>
<td>556</td>
<td>2 856</td>
<td>14 122</td>
<td>68 927</td>
<td>334 632</td>
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<td>$\mathcal{N}^{BKL+}_{5,d}$</td>
<td>42</td>
<td>773</td>
<td>11 124</td>
<td>147 855</td>
<td>1 917 046</td>
<td>24 672 817</td>
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<td>$\mathcal{N}^{BKL+}_{6,d}$</td>
<td>132</td>
<td>6 743</td>
<td>266 944</td>
<td>9 845 829</td>
<td>356 470 124</td>
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</tr>
</tbody>
</table>

### Questions about columns (OK for $d \leq 2$):

- What is the behaviour of $\mathcal{N}^{BKL+}_{n,3}$, etc.?

### Questions about rows (OK for $n \leq 3$):

- Can one reduce the size of $\mathcal{M}_n^*$?
- Is $\mathcal{M}_n^*$ always invertible?
- What is the asymptotic behaviour of the spectral radius of $\mathcal{M}_n^*$?

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$\text{tr}(\mathcal{M}_n^*)$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
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<td>$\text{det}(\mathcal{M}_n^*)$</td>
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<td>1</td>
<td>2</td>
<td>2$^4$.5</td>
<td>2$^{16}$.5$^5$.7</td>
<td>2$^{63}$.3$^5$.2$^{11}$.7$^7$</td>
<td>2$^{247}$.3$^8$.5$^8$.4$^{35}$.11</td>
</tr>
<tr>
<td>$\rho(\mathcal{M}_n^*)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4.83...</td>
<td>12.83...</td>
<td>35.98...</td>
<td>104.87...</td>
</tr>
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</table>
• Whenever a group admits a finite Garside structure, there is a finite state automaton, whence an incidence matrix.

• The associated combinatorics is likely to be interesting if the Garside structure is connected with combinatorially meaningful objects: permutations (Garside case), noncrossing partitions (Birman–Ko–Lee case), etc.

• The family of group(oid)s that admit an interesting Garside structure is large and so far not well understood:
  ▶ for instance (Bessis, 2006) free groups do;
  ▶ also: exotic Garside structures on braid groups;
  ▶ and exotic non-Garside normal forms with local characterizations;
  ▶ most results involving braids extend to Artin–Tits groups of spherical type (i.e., associated with a finite Coxeter group);
  ▶ many potential combinatorial problems

• Specific case of dual braid monoids and noncrossing partitions:
  ▶ (almost) nothing known so far,
  ▶ but the analogy $B_n^*/B_n^+$ suggests that combinatorics could be interesting (?).
• J. Mairesse & F. Matheus, Growth series for Artin groups of dihedral type

• P. Dehornoy, Combinatorics of normal sequences of braids,

• F. Hivert, J.-C. Novelli, J.-Y. Thibon, Sur une conjecture de Dehornoy

• P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, J. Michel, Foundations of

• L. Carlucci, P. Dehornoy, A. Weiermann, Unprovability statements involving braids;

• D. Bessis, F. Digne, J. Michel, Springer theory in braid groups and the
  Birman-Ko-Lee monoid;

www.math.unicaen.fr/~dehornoy