Braid combinatorics, permutations, and noncrossing partitions
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- A few combinatorial questions involving braids and their Garside structures:
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A few combinatorial questions involving braids and their Garside structures:
  the classical Garside structure, connected with permutations,
  the dual Garside structure, connected with noncrossing partitions.
Plan:
• Plan:

  1. Braid combinatorics: Artin generators
Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
• Plan:
  1. Braid combinatorics: Artin generators
  2. Braid combinatorics: Garside generators
- Plan:
  1. Braid combinatorics: Artin generators
  2. Braid combinatorics: Garside generators
Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
• a 4-strand braid diagram
• a 4-strand braid diagram
• a 4-strand braid diagram = 2D-projection of a 3D-figure:
• a 4-strand braid diagram  = 2D-projection of a 3D-figure:
• a 4-strand braid diagram  = 2D-projection of a 3D-figure:

• isotopy  = move the strands but keep the ends fixed:
• a 4-strand braid diagram

= 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:
• a 4-strand braid diagram

= 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:

isotopic to
• a 4-strand braid diagram

= 2D-projection of a 3D-figure:

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• a 4-strand braid diagram = 2D-projection of a 3D-figure:

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- isotopy = move the strands but keep the ends fixed:

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• a 4-strand braid diagram = 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:
• a 4-strand **braid diagram**  

\[= \text{2D-projection of a 3D-figure:}\]

\[\text{←←←} \quad \text{isotopy} = \text{move the strands but keep the ends fixed:} \]

\[\text{isotopic to} \quad \text{isotopic to}\]
• a 4-strand braid diagram = 2D-projection of a 3D-figure:

\[ \text{isotopy} = \text{move the strands but keep the ends fixed:} \]

\[ \text{isotopic to} \]
• a 4-strand braid diagram

= 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:
• a 4-strand braid diagram = 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:

isotopic to
- a 4-strand braid diagram = 2D-projection of a 3D-figure:

- isotopy = move the strands but keep the ends fixed:

- a braid := an isotopy class represented by 2D-diagram,
• a 4-strand braid diagram = 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:

• a braid := an isotopy class represented by 2D-diagram, but different 2D-diagrams may give rise to the same braid.
• Product of two braids:
- **Product** of two braids:
• **Product** of two braids:
• **Product** of two braids:

```
  \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \end{array} \quad \star \quad \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \end{array} \quad := \quad \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \end{array}
```

• Then well-defined \textit{(with respect to isotopy)}, associative, admits a unit:

```
  \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \end{array} \quad \star \quad \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \end{array} \quad = \quad \begin{array}{ccc}
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots \\
  \end{array}
```
• **Product** of two braids:

![Diagram of two braids being multiplied](image)

• Then well-defined *(with respect to isotopy)*, associative, admits a unit:

![Diagram of a braid being multiplied by the unit](image)
• **Product** of two braids:

\[ \star \quad := \]

• Then well-defined (with respect to isotopy), associative, admits a unit:

\[ \star \quad = \quad \simeq \]
- **Product** of two braids:

\[ \begin{array}{ccc}
\text{Braid 1} & \times & \text{Braid 2} \\
\text{Definition} & : & \text{Result}
\end{array} \]

- Then well-defined (with respect to isotopy), associative, admits a unit:

\[ \begin{array}{ccc}
\text{Braid 1} & \times & \text{Braid 2} \\
\text{Result} & : & \text{Isotopic to}
\end{array} \]
- **Product** of two braids:

  ![Product of two braids](image)

  - Then well-defined *(with respect to isotopy)*, associative, admits a unit:

    ![Well-defined, associative, unit](image)

    and inverses:

    ![Inverses](image)

  `braid`
• **Product** of two braids:

\[
\begin{array}{c}
\text{blue braid} \\
\otimes \\
\text{orange braid} \\
\text{blue braid}
\end{array}
\]

Then well-defined (with respect to isotopy), associative, admits a unit:

\[
\begin{array}{c}
\text{blue braid} \\
\otimes \\
\text{braid} \\
\text{blue braid}
\end{array}
\]

and inverses:

\[
\begin{array}{c}
\text{blue braid} \\
\otimes \\
\text{braid} \\
\text{blue braid}
\end{array}
\]

isotopic to
• **Product** of two braids:

\[ \begin{array}{ccc}
\text{blue braid} & \ast & \text{orange braid} \\
\text{blue braid} & = & \text{orange braid}
\end{array} \]

Then well-defined *(with respect to isotopy)*, associative, admits a unit:

\[ \begin{array}{ccc}
\text{blue braid} & \ast & \text{blue braid} \\
\text{blue braid} & = & \text{blue braid}
\end{array} \approx \text{blue braid} \uparrow \text{isolotopic to}

and inverses:
• **Product** of two braids:

\[
\begin{align*}
\text{blue braid} \ast \text{orange braid} & := \text{blue braid} \\
\end{align*}
\]

• Then well-defined (with respect to isotopy), associative, admits a unit:

\[
\begin{align*}
\text{blue braid} \ast \text{blue braid} & = \text{blue braid} & & \text{isotopic to} \\
\end{align*}
\]

and inverses:

\[
\begin{align*}
\text{blue braid} \ast \text{blue braid} & = \text{blue braid} \\
\end{align*}
\]
- **Product** of two braids:

    \[
    \begin{array}{c}
    \text{Product} \\
    \end{array} 
    \hspace{1cm}
    \begin{array}{c}
    \text{Product} \\
    \end{array} 
    \hspace{1cm}
    \begin{array}{c}
    \text{Product} \\
    \end{array}
    \]

- Then well-defined (with respect to isotopy), associative, admits a unit:

    \[
    \begin{array}{c}
    \text{Product} \\
    \end{array} 
    \hspace{1cm}
    \begin{array}{c}
    \text{Product} \\
    \end{array} 
    \hspace{1cm}
    \begin{array}{c}
    \text{Product} \\
    \end{array}
    \]

- and inverses:

    \[
    \begin{array}{c}
    \text{Product} \\
    \end{array} 
    \hspace{1cm}
    \begin{array}{c}
    \text{Product} \\
    \end{array} 
    \hspace{1cm}
    \begin{array}{c}
    \text{Product} \\
    \end{array}
    \]
- **Product** of two braids:

\[
\begin{array}{c}
\text{braid} \\
\times \\
\text{braid} \\
\end{array} := \begin{array}{c}
\text{braid} \\
\text{braid} \\
\end{array}
\]

- Then well-defined (with respect to isotopy), associative, admits a unit:

\[
\begin{array}{c}
\text{braid} \\
\times \\
\text{braid} \\
\end{array} \approx \begin{array}{c}
\text{braid} \\
\text{braid} \\
\end{array}
\]

and inverses:

\[
\begin{array}{c}
\text{braid} \\
\times \\
\text{braid} \\
\end{array} \approx \begin{array}{c}
\text{braid} \\
\text{braid} \\
\end{array}
\]

- For each \( n \), the group \( B_n \) of \( n \)-strand braids (E. Artin, 1925).
• Artin generators of $B_n$: 

[Diagram of Artin generators]
• Artin generators of $B_n$:
Artin presentation of $B_n$

- Artin generators of $B_n$:
• Artin generators of $B_n$: 

\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} \\
\end{align*}
Artin presentation of $B_n$:

- Artin generators of $B_n$:

\[ \sigma_1 \]
• Artin generators of $B_n$:
• Artin generators of $B_n$: 

\[ \sigma_1 \quad \sigma_2 \quad \sigma_3 \]
• Artin generators of $B_n$:
• Artin generators of $B_n$:

\[
\sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1}
\]

• **Theorem (Artin).**— The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$,
- Artin generators of $B_n$:

\[
\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_1^{-1}
\]

- **Theorem (Artin).**— The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

\[
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1,
\]
• Artin generators of $B_n$:

\[
\begin{array}{c}
\sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1} \\
= \\
\begin{array}{cccc}
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_1^{-1}
\end{array}
\end{array}
\]

• **Theorem (Artin).**— The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

\[
\begin{align*}
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| \geq 2.
\end{align*}
\]
Artin presentation of $B_n$:

- Artin generators of $B_n$:

\[
\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2
\]

- Theorem (Artin).— The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

\[
\begin{align*}
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| \geq 2.
\end{align*}
\]
Artin presentation of $B_n$

- Artin generators of $B_n$:

$$\sigma_1 \sigma_2 \sigma_1 \approx \sigma_2 \sigma_1 \sigma_2$$

- Theorem (Artin).— The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

$$\begin{cases} 
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
\sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i - j| \geq 2.
\end{cases}$$
• For $n \geq 2$, the group $B_n$ is infinite ➔ consider finite subsets.
• For \( n \geq 2 \), the group \( B_n \) is infinite ★ consider finite subsets.

• \( B_n^+ := \) monoid of classes of \( n \)-strand positive diagrams
• For $n \geq 2$, the group $B_n$ is infinite ➤ consider finite subsets.

• $B^+_n :=$ monoid of classes of $n$-strand positive diagrams
  
  all crossings have a positive orientation
• For $n \geq 2$, the group $B_n$ is infinite ✔ consider finite subsets.

• $B_n^+:=$ monoid of classes of $n$-strand positive diagrams
  all crossings have a positive orientation

• Theorem (Garside, 1967).— As a monoid, $B_n^+$ admits the presentation... (as $B_n$);
  it is cancellative, and admits lcms and gcds.
• For $n \geq 2$, the group $B_n$ is infinite. Consider finite subsets.

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• Hence: Equivalent positive braid words have the same length,
• For $n \geq 2$, the group $B_n$ is infinite \(\Rightarrow\) consider finite subsets.

• $B_n^+ :=$ monoid of classes of $n$-strand positive diagrams
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• **Theorem (Garside, 1967).** — As a monoid, $B_n^+$ admits the presentation... (as $B_n$); it is cancellative, and admits lcms and gcds.

• Hence: Equivalent positive braid words have the same length,
  \(\Rightarrow\) every positive braid $\beta$ has a well-defined length $\|\beta\|^\text{Art}$ w.r.t. Artin generators $\sigma_i$. 
• For $n \geq 2$, the group $B_n$ is infinite ➤ consider finite subsets.

• $B_n^+:=$ monoid of classes of $n$-strand positive diagrams

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• **Theorem (Garside, 1967).**— As a monoid, $B_n^+$ admits the presentation... (as $B_n$); it is cancellative, and admits lcms and gcds.

• Hence: Equivalent positive braid words have the same length,
  ➤ every positive braid $\beta$ has a well-defined length $\|\beta\|^\text{Art}$ w.r.t. Artin generators $\sigma_i$.

• **Question:** Determine $N_{n,\ell}^{\text{Art}+} := \#\{\beta \in B_n^+ \mid \|\beta\|^\text{Art} = \ell\}$
• For \( n \geq 2 \), the group \( B_n \) is infinite \( \implies \) consider finite subsets.

\[ B^+_{\infty} := \text{monoid of classes of } n\text{-strand positive diagrams} \]
\[ \text{all crossings have a positive orientation} \]

\[ \text{Theorem (Garside, 1967).— As a monoid, } B^+_{\infty} \text{ admits the presentation... (as } B_n) ; \]
\[ \text{it is cancellative, and admits lcms and gcds.} \]

• Hence: Equivalent positive braid words have the same length,
\[ \implies \text{every positive braid } \beta \text{ has a well-defined length } || \beta ||^{\text{Art}} \text{ w.r.t. Artin generators } \sigma_i. \]

• Question: Determine \( N^{\text{Art} +}_{n, \ell} := \# \{ \beta \in B^+_{\infty} \mid || \beta ||^{\text{Art}} = \ell \} \)
\[ \text{and/or the associated generating series.} \]
Theorem (Deligne, 1972).— For every $n$, the g.f. of $N_{n,\ell}^{\text{Art}+}$ is rational.

Proof: For $\beta$ in $B_n^+$, define $M(\beta) := \{\beta \gamma \mid \gamma \in B_n^+\} = $ right-multiples of $\beta$. 
• **Theorem (Deligne, 1972).**— For every \( n \), the g.f. of \( N^{\text{Art}+}_{n, \ell} \) is rational.

• Proof: For \( \beta \) in \( B^+_n \), define \( M(\beta) := \{ \beta \gamma | \gamma \in B^+_n \} = \text{right-multiples of } \beta \).
  
  ▶ Then \( B^+_n \setminus \{1\} = \bigcup_i M(\sigma_i) \), and
Theorem (Deligne, 1972).— For every $n$, the g.f. of $N^{Art+}_{n,\ell}$ is rational.

Proof: For $\beta$ in $B^+_n$, define $M(\beta) := \{\beta \gamma \mid \gamma \in B^+_n\} = \text{right-multiples of } \beta$. Then $B^+_n \setminus \{1\} = \bigcup_i M(\sigma_i)$, and $M(\sigma_i) \cap M(\sigma_j) = M(\text{lcm}(\sigma_i, \sigma_j))$. 
• **Theorem (Deligne, 1972).**— For every $n$, the g.f. of $N_{n,\ell}^{\text{Art}+}$ is rational.

• **Proof:** For $\beta$ in $B_n^+$, define $M(\beta) := \{ \beta \gamma \mid \gamma \in B_n^+ \} =$ right-multiples of $\beta$.
  
  ▶ Then $B_n^+ \setminus \{1\} = \bigcup_i M(\sigma_i)$, and $M(\sigma_i) \cap M(\sigma_j) = M(\text{lcm}(\sigma_i, \sigma_j))$.
  
  ▶ By inclusion–exclusion, get induction $N_{n,\ell}^{\text{Art}+} = c_1 N_{n,\ell-1}^{\text{Art}+} + \cdots + c_K N_{n,\ell-K}^{\text{Art}+}$. $\square$
• **Theorem (Deligne, 1972).**— For every $n$, the g.f. of $N_{n,\ell}^{\text{Art}+}$ is rational.

• Proof: For $\beta$ in $B^+_n$, define $M(\beta):=\{\beta \gamma \mid \gamma \in B^+_n\} = \text{right-multiples of } \beta$.
  
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• More precisely: for every $n$, the generating series of $N_{n,\ell}^{\text{Art}+}$ is the inverse of a polynomial $P_n(t)$. 
Theorem (Deligne, 1972).— For every $n$, the g.f. of $N_{n,\ell}^{\text{Art}+}$ is rational.

Proof: For $\beta$ in $B_n^+$, define $M(\beta) := \{ \beta \gamma \mid \gamma \in B_n^+ \} =$ right-multiples of $\beta$.

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- By inclusion–exclusion, get induction $N_{n,\ell}^{\text{Art}+} = c_1 N_{n,\ell-1}^{\text{Art}+} + \cdots + c_K N_{n,\ell-K}^{\text{Art}+}$. □

More precisely: for every $n$, the generating series of $N_{n,\ell}^{\text{Art}+}$ is the inverse of a polynomial $P_n(t)$.

Proposition (Bronfman, 2001).— Starting from $P_0(t) = P_1(t) = 1$, one has

$$P_n(t) = \sum_{i=1}^n (-1)^{i+1} t^{i(i-1)/2} P_{n-i}(t).$$
• **Theorem (Deligne, 1972).**— For every \( n \), the g.f. of \( N_{n, \ell}^{\text{Art}+} \) is rational.

Proof: For \( \beta \) in \( B_n^+ \), define \( M(\beta) := \{ \beta \gamma \mid \gamma \in B_n^+ \} = \text{right-multiples of } \beta \).
- Then \( B_n^+ \setminus \{1\} = \bigcup_i M(\sigma_i) \), and \( M(\sigma_i) \cap M(\sigma_j) = M(\text{lcm}(\sigma_i, \sigma_j)) \).
- By inclusion–exclusion, get induction \( N_{n, \ell}^{\text{Art}+} = c_1 N_{n, \ell-1}^{\text{Art}+} + \cdots + c_K N_{n, \ell-K}^{\text{Art}+} \).

More precisely: for every \( n \), the generating series of \( N_{n, \ell}^{\text{Art}+} \) is the inverse of a polynomial \( P_n(t) \).

**Proposition (Bronfman, 2001).**— Starting from \( P_0(t) = P_1(t) = 1 \), one has

\[
P_n(t) = \sum_{i=1}^{n} (-1)^{i+1} t^{i(i-1)/2} P_{n-i}(t).
\]
• Same question for $B_n$ instead of $B_n^+$;
• Same question for $B_n$ instead of $B_n^+$; all representatives don't have the same length
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  ▶ define $\|\beta\|^{\text{Art}} :=$ the minimal length of a word representing $\beta$. 
• Same question for $B_n$ instead of $B_n^+$; all representatives don't have the same length
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• **Question:** Determine $N_{n,\ell}^{\text{Art}} := \#\{\beta \in B_n \mid \|\beta\|^{\text{Art}} = \ell\}$
  and/or determine the associated generating series.
- Same question for $B_n$ instead of $B^+_n$; all representatives don’t have the same length
  - define $\|\beta\|^{Art} :=$ the minimal length of a word representing $\beta$.

- **Question:** Determine $N_{n,\ell}^{Art} := \#\{\beta \in B_n \mid \|\beta\|^{Art} = \ell\}$ and/or determine the associated generating series.

- **Proposition (Mairesse–Matheus, 2005).** The generating series of $N_{3,\ell}^{Art}$ is
  
  $$1 + \frac{2t(2 - 2t - t^2)}{(1-t)(1-2t)(1-t-t^2)}. $$
• Same question for $B_n$ instead of $B_n^+$; all representatives don’t have the same length
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• Then open, even $N_{4,\ell}^{\text{Art}}$:
• Same question for $B_n$ instead of $B_n^+$; all representatives don’t have the same length
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• Then open, even $N_{4,\ell}^{Art}$: (Mairesse) no rational fraction with degree $\leq 13$ denominator.
• Same question for $B_n$ instead of $B_n^+$; all representatives don’t have the same length
  
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• “Explanation”: Artin generators are not the right generators...
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• Then open, even $N_{4,\ell}^{\text{Art}}$ : (Mairesse) no rational fraction with degree $\leq 13$ denominator.

• “Explanation”: Artin generators are not the right generators...
  ▶ change generators
Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
• **Definition:** A Garside structure in a group $G$ is a subset $S$ of $G$ s.t. every element $g$ of $G$ admits an $S$-normal decomposition,
Definition: A Garside structure in a group $G$ is a subset $S$ of $G$ s.t. every element $g$ of $G$ admits an $S$-normal decomposition, meaning $g = s_p^{-1} \cdots s_1^{-1} t_1 \cdots t_q$ with $s_1, \ldots, s_p, t_1, \ldots, t_q$ in $S$ and, using “$f$ left-divides $g$” for “$f^{-1}g$ lies in the submonoid $\hat{S}$ of $G$ generated by $S$”,
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Garside structure

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```
1 4
3
2
1
```
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```
3  4
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  \[
  \begin{array}{c}
  3 & 4 \\
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  4 & 1 \\
  \end{array}
  \rightarrow (4, 2, 1, 3)
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  ![Braid Diagram]

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\[(4, 2, 1, 3) \mapsto \]

```
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  ▶ The family \( S_n \) of all simple \( n \)-strand braids is a copy of \( S_n \).
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[Diagram of braids]

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• A new family of generators: the Garside generators $\sigma_i$
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• “Garside’s fundamental braid” \( \Delta_n := \sigma_{(n,\ldots,1)} \), whence \( \Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1 \):
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  \Delta_1 = 1, \quad \Delta_2 = \sigma_1, \quad \Delta_3 = \sigma_1 \sigma_2 \sigma_1, \quad \text{etc.}
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  ▶ the poset $(S_n, \triangleleft)$ is isomorphic to $(\mathcal{S}_n, \triangleleft)$.

  left-divisibility in $B_n^+$ weak order in $\mathcal{S}_n$
• **Question:** Determine \( N_{n,\ell}^{\text{Gar}^+} := \# \{ \beta \in B_n^+ \mid \|\beta\|^{\text{Gar}} = \ell \} \) and/or its generating series, where \( \|\beta\|^{\text{Gar}} := \text{length of the } S_n\text{-normal decomposition.} \)
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• **Proposition.**— Let $M_n$ be the $n! \times n!$ matrix indexed by simple braids *(i.e., by permutations)* s.t. \((M_n)_{s,t} = \begin{cases} 1 & \text{if } (s, t) \text{ is normal}, \\ 0 & \text{otherwise}. \end{cases}\)
• **Question:** Determine $N^\text{Gar}_n,\ell := \#\{\beta \in B^+_n \mid \|\beta\|^\text{Gar} = \ell\}$ and/or its generating series, where $\|\beta\|^\text{Gar} := \text{length of the } S_n\text{-normal decomposition.}$

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Then $N^\text{Gar}_n,\ell$ is the $\text{idth entry in } (1, \ldots, 1) \cdot M^\ell_n$. 
• **Question:** Determine $N_{n, \ell}^{\text{Gar}+} := \# \{ \beta \in B_n^+ \mid \| \beta \|_{\text{Gar}} = \ell \}$ and/or its generating series, where $\| \beta \|_{\text{Gar}} := \text{length of the } S_n\text{-normal decomposition.}$

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Then $N_{n, \ell}^{\text{Gar}+}$ is the idth entry in $(1, \ldots, 1) \cdot M_n^\ell.$

▶ For each $n,$ the generating series of $N_{n, \ell}^{\text{Gar}+}$ is rational.
Lemma 1: For $f, g$ in $\mathcal{S}_n$, the pair $(\sigma_f, \sigma_g)$ is normal iff $\text{Desc}(f) \supseteq \text{Desc}(g^{-1})$.

$\uparrow$

descents of $f := \{k | f(k) > f(k + 1)\}$
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Reducing the size of the matrix

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- **Lemma 2**: The \# of permutations \( f \) satisfying \( \text{Desc}(f) \supseteq I \) and \( \text{Desc}(f^{-1}) \supseteq J \) is the \# of \( k \times \ell \) matrices with entries in \( \mathbb{N} \) s.t. the sum of the \( i \)th row is \( p_i \) and the sum of the \( j \)th column is \( q_j \), with \( (p_1, \ldots, p_k) \) the composition of \( I \) and \( (q_1, \ldots, q_\ell) \) that of \( J \).
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- Remarks:
  - Going from $M_n$ to $M''_n \approx$ reducing the size of the automatic structure of $B_n$
    from $n!$ to $p(n)$ ($\sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2n/3}}$)
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• Hence \( (M'_n)_{I,J} \) only depends on the partition of \( J \);
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• Remarks:
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  from \( n! \) to \( p(n) \) (\( \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2n/3}} \))
  ► \text{(Hohlweg)} That \( (M'_n)_{I,J} \) only depends on the partition of \( J \) is
  (another) form of Solomon’s result about the descent algebra.
• The growth rate of $N_{n,\ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M''_n$: 
• The growth rate of $N_{n,\ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M_n^{''}$: 
\[ \text{CharPol}(M_1^{''}) = x - 1 \]
The growth rate of $N_{n, \ell}^{\text{Gar}+}$ is connected with the eigenvalues of $M_n$, hence of $M''_n$:

\[
\text{CharPol}(M''_1) = x - 1
\]
\[
\text{CharPol}(M''_2) = \text{CharPol}(M''_1) \cdot (x - 1)
\]
• The growth rate of $N_{n, \ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M_n''$:

\[
\begin{align*}
\text{CharPol}(M_1'') &= x - 1 \\
\text{CharPol}(M_2'') &= \text{CharPol}(M_1'') \cdot (x - 1) \\
\text{CharPol}(M_3'') &= \text{CharPol}(M_2'') \cdot (x - 2)
\end{align*}
\]
• The growth rate of $N_{n,\ell}^{Gar+}$ is connected with the eigenvalues of $M_n$, hence of $M''_n$:

\[
\begin{align*}
\text{CharPol}(M''_1) &= x - 1 \\
\text{CharPol}(M''_2) &= \text{CharPol}(M''_1) \cdot (x - 1) \\
\text{CharPol}(M''_3) &= \text{CharPol}(M''_2) \cdot (x - 2) \\
\text{CharPol}(M''_4) &= \text{CharPol}(M''_3) \cdot (x^2 - 6x + 3)
\end{align*}
\]
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\text{CharPol}(M''_4) &= \text{CharPol}(M''_3) \cdot (x^2 - 6x + 3) \\
\text{CharPol}(M''_5) &= \text{CharPol}(M''_4) \cdot (x^2 - 20x + 24), \\
\end{align*}
• The growth rate of $N_{n+1}^{Gar+}$ is connected with the eigenvalues of $M_n$, hence of $M''_n$:

\[
\begin{align*}
\text{CharPol}(M''_1) &= x - 1 \\
\text{CharPol}(M''_2) &= \text{CharPol}(M''_1) \cdot (x - 1) \\
\text{CharPol}(M''_3) &= \text{CharPol}(M''_2) \cdot (x - 2) \\
\text{CharPol}(M''_4) &= \text{CharPol}(M''_3) \cdot (x^2 - 6x + 3) \\
\text{CharPol}(M''_5) &= \text{CharPol}(M''_4) \cdot (x^2 - 20x + 24),...
\end{align*}
\]

• **Theorem (Hivert–Novelli–Thibon).** —

The characteristic polynomial of $M''_n$ divides that of $M''_{n+1}$.
• The growth rate of $N_{n,k}^{\text{Gar}+}$ is connected with the eigenvalues of $M_n$, hence of $M''_n$:

\[
\begin{align*}
\text{CharPol}(M''_1) &= x - 1 \\
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\text{CharPol}(M''_3) &= \text{CharPol}(M''_2) \cdot (x - 2) \\
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\end{align*}
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• **Theorem (Hivert–Novelli–Thibon).**—

The characteristic polynomial of $M''_n$ divides that of $M''_{n+1}$.

- **Proof:** Interpret $M''_n$ in terms of quasi-symmetric functions in the sense of Malvenuto–Reutenauer, and determine the LU-decomposition.
The growth rate of $N_{n, \ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M_n''$:

- $\text{CharPol}(M_1'') = x - 1$
- $\text{CharPol}(M_2'') = \text{CharPol}(M_1'') \cdot (x - 1)$
- $\text{CharPol}(M_3'') = \text{CharPol}(M_2'') \cdot (x - 2)$
- $\text{CharPol}(M_4'') = \text{CharPol}(M_3'') \cdot (x^2 - 6x + 3)$
- $\text{CharPol}(M_5'') = \text{CharPol}(M_4'') \cdot (x^2 - 20x + 24),...$

**Theorem (Hivert–Novelli–Thibon).—**

The characteristic polynomial of $M_n''$ divides that of $M_{n+1}''$.

**Proof:** Interpret $M_n''$ in terms of quasi-symmetric functions in the sense of Malvenuto–Reutenauer, and determine the LU-decomposition.

- Spectral radius:
• The growth rate of $N_{n,\ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M_n''$:  
\[
\begin{align*}
\text{CharPol}(M_1'') &= x - 1 \\
\text{CharPol}(M_2'') &= \text{CharPol}(M_1'') \cdot (x - 1) \\
\text{CharPol}(M_3'') &= \text{CharPol}(M_2'') \cdot (x - 2) \\
\text{CharPol}(M_4'') &= \text{CharPol}(M_3'') \cdot (x^2 - 6x + 3) \\
\text{CharPol}(M_5'') &= \text{CharPol}(M_4'') \cdot (x^2 - 20x + 24),...
\end{align*}
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• **Theorem (Hivert–Novelli–Thibon).**— The characteristic polynomial of $M_n''$ divides that of $M_{n+1}''$.

  ▶ Proof: Interpret $M_n''$ in terms of quasi-symmetric functions in the sense of Malvenuto–Reutenauer, and determine the LU-decomposition.  

• **Spectral radius:**

<table>
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<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(M_n)$</td>
<td>1</td>
<td>2</td>
<td>5.5</td>
<td>18.7</td>
<td>77.4</td>
<td>373.9</td>
<td>2066.6</td>
</tr>
</tbody>
</table>
The growth rate of $N_{n,\ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M''_n$:

\[
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\- **Theorem (Hivert–Novelli–Thibon).**—

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\- **Spectral radius:**

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\rho(M_n) & 1 & 2 & 5.5 & 18.7 & 77.4 & 373.9 & 2066.6 \\
\hline
\rho(M_n) / (n \rho(M_{n-1})) & 0.5 & 0.667 & 0.681 & 0.687 & 0.689 & 0.690 & 0.691 \\
\hline
\end{array}
\]
• The growth rate of $N_{n,\ell}^{\text{Gar}^+}$ is connected with the eigenvalues of $M_n$, hence of $M_n''$:

CharPol($M_1''$) = $x - 1$
CharPol($M_2''$) = CharPol($M_1''$) · ($x - 1$)
CharPol($M_3''$) = CharPol($M_2''$) · ($x - 2$)
CharPol($M_4''$) = CharPol($M_3''$) · ($x^2 - 6x + 3$)
CharPol($M_5''$) = CharPol($M_4''$) · ($x^2 - 20x + 24$),...

• Theorem (Hivert–Novelli–Thibon).—

The characteristic polynomial of $M_n''$ divides that of $M_{n+1}''$.

Proof: Interpret $M_n''$ in terms of quasi-symmetric functions in the sense of Malvenuto–Reutenauer, and determine the LU-decomposition.

• Spectral radius:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
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</tr>
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What is the asymptotic behaviour?
• So far: $N_{n,\ell}^{\text{Gar}^+}$ with $n$ fixed and $\ell$ varying;
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• Proposition. — $N_{n,2}^{\text{Gar}^+} = \sum_{0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 N_{i,2}^{\text{Gar}^+}$, 
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**Proposition.**

$N_{n,2}^{\text{Gar}^+} = \sum_{i=0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 N_{i,2}^{\text{Gar}^+}$,

whence (Carlitz–Scoville–Vaughan) $1 + \sum_n N_{n,2}^{\text{Gar}^+} \frac{z^n}{(n!)^2} = \frac{1}{J_0(\sqrt{z})}$. 

Bessel function $J_0$
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**Put**  
\( N_{n,\ell}^{\text{Gar}^+}(s) := \# \text{ normal sequences in } B_n^+ \text{ finishing with } s \):  
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• Conclusion: Braid combinatorics w.r.t. Garside generators
  leads to new, interesting (?) questions about permutation combinatorics.
Motivation

- Braid groups are countable, braids can be encoded in integers, and most of their (algebraic) properties can be proved in the logical framework of Peano arithmetic, and even of weaker subsystems, like $I\Sigma_1$ where induction is limited to formulas involving at most one unbounded quantifier.
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- **Proof:** Evaluate $\#\{\beta \in B^+_3 \mid \|\beta\|^{Gar} \leq \ell \& \beta < \Delta^k_3\}$.  

□
• Plan:

1. Braid combinatorics: Artin generators
2. Braid combinatorics: Garside generators
Another family of generators for $B_n$: the Birman–Ko–Lee generators
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$$a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}^{-1}$$

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\[ \begin{array}{c}
\text{i} & \text{j} \\
\text{=}\end{array} \Rightarrow \begin{array}{c}
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![Diagram](attachment:image.png)
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• The dual braid monoid: the submonoid $B_n^{++}$ of $B_n$ generated by the elements $a_{i,j}$.
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7 & 8 \\
\end{array} \]
• **Proposition (Bessis–Digne–Michel).**— The elements of the Garside structure $S_n^*$ (divisors of $\delta_n$ in $B_n^+^*$) are the elements $a_P$ with $P$ a union of disjoint polygons with $n$ vertices, hence in 1-1 correspondence with the $\text{Cat}_n$ noncrossing partitions of $\{1, \ldots, n\}$.

- notation $a_\lambda$ for $\lambda$ a noncrossing partition

- Examples:

  - $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\} \leftrightarrow a_{2,8}a_{3,5}a_{5,6}$
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**Examples:**

- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\} \leftrightarrow \begin{array}{c} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \\ \begin{array}{c} \{2, 8\} \\ \{3, 5, 6\} \\ \{4\} \\ \{7\} \end{array} \end{array} \leftrightarrow a_{2,8} a_{3,5} a_{5,6}$

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- $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\} \leftrightarrow \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8
\end{array} \leftrightarrow a_{2,8} a_{3,5} a_{5,6}$

- $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\} \leftrightarrow \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8
\end{array} \leftrightarrow \delta_8 = a_{12} a_{23} \cdots a_{78}$
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  ▶ notation $a_\lambda$ for $\lambda$ a noncrossing partition

• Examples:

  ▶ $\{\{1\}, \{2, 8\}, \{3, 5, 6\}, \{4\}, \{7\}\} \leftrightarrow \begin{array}{c}
\begin{tikzpicture}
\foreach \x in {1,2,3,4,5,6,7,8}
\node at (90 - \x * 45:1) {$\x$};
\draw (1) -- (2); \draw (2) -- (3); \draw (3) -- (4); \draw (4) -- (5); \draw (5) -- (6); \draw (6) -- (7); \draw (7) -- (8); \draw (8) -- (1);
\end{tikzpicture}
\end{array} \leftrightarrow a_{2,8} a_{3,5} a_{5,6}$

  ▶ $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\} \leftrightarrow 7 \leftrightarrow \delta_8 = a_{12} a_{23} \cdots a_{78}$

• Remark: The permutation of the braid $a_\lambda$ is the permutation associated with $\lambda$ (product of cycles of the parts)
• **Question:** Determine $N_{n,\ell}^{B KL} := \#\{\beta \in B_n^+ | \|\beta\|^B KL = \ell\}$ and its generating series, where $\|\beta\|^B KL :=$ length of the $S_n^*$-normal decomposition of $\beta$. 
• **Question:** Determine $N_{n,\ell}^{BKL^+} := \#\{\beta \in B_n^+ | \|\beta\|^BKL = \ell\}$ and its generating series, where $\|\beta\|^BKL :=$ length of the $S_n^*$-normal decomposition of $\beta$.

• For instance: $N_{n,1}^{BKL^+} = \#S_n^* = \text{Cat}_n$. 
• **Question:** Determine \( N_{n, \ell}^{BKL^+} := \#\{ \beta \in B_n^+ | \|\beta\|^{BKL} = \ell \} \) and its generating series, where \( \|\beta\|^{BKL} := \) length of the \( S_n^* \)-normal decomposition of \( \beta \).

• For instance: \( N_{n, 1}^{BKL^+} = \# S_n^* = \text{Cat}_n \).

• Exactly similar to the classical case: local property, etc.
• **Question**: Determine $N_{n, \ell}^{BKL} := \# \{ \beta \in B_n^+ \mid \|\beta\|^{BKL} = \ell \}$ and its generating series, where $\|\beta\|^{BKL} := \text{length of the } S_n^* \text{-normal decomposition of } \beta$.

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---

• **Proposition.**— Let $M_n^*$ be the $\text{Cat}_n \times \text{Cat}_n$ matrix indexed by noncrossing partitions s.t. 
\[
(M_n^*)_{\lambda, \mu} = \begin{cases} 
1 & \text{if } (a_\lambda, a_\mu) \text{ is } S_n^* \text{-normal}, \\
0 & \text{otherwise.}
\end{cases}
\]
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• **Proposition.**— Let \( M_n^* \) be the \( \text{Cat}_n \times \text{Cat}_n \) matrix indexed by noncrossing partitions s.t. \( (M_n^*)_{\lambda,\mu} = \begin{cases} 1 & \text{if } (a_{\lambda}, a_{\mu}) \text{ is } S_n^*\text{-normal}, \\ 0 & \text{otherwise.} \end{cases} \) Then \( N_{n,\ell}^{BKL} \) is the \( 1_n \text{th entry in } (1, \ldots, 1) \cdot M_n^* \ell. \)
• **Question:** Determine $N_{n,\ell}^{\text{BKL}+} := \#\{\beta \in B_{n}^+ \mid \|\beta\|^{\text{BKL}} = \ell\}$ and its generating series, where $\|\beta\|^{\text{BKL}} := \text{length of the } S_n^*\text{-normal decomposition of } \beta$.

• For instance: $N_{n,1}^{\text{BKL}+} = \#S_n^* = \text{Cat}_n$.

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• **Proposition.**— Let $M_n^*$ be the $\text{Cat}_n \times \text{Cat}_n$ matrix indexed by noncrossing partitions s.t. $\left(M_n^*\right)_{\lambda,\mu} = \begin{cases} 1 & \text{if } (a_\lambda, a_\mu) \text{ is } S_n^*\text{-normal}, \\ 0 & \text{otherwise}. \end{cases}$ Then $N_{n,\ell}^{\text{BKL}+}$ is the $1_n$th entry in $(1, \ldots, 1) \cdot M_n^* \ell$.

  • For every $n$, the generating series of $N_{n,\ell}^{\text{BKL}+}$ is rational.
• When is \((a_\lambda, a_\mu)\) \(S_n^*\)-normal?
The normality relation

- When is \((a_\lambda, a_\mu) \ S^*_n\)-normal?

- Recall: If a Garside structure \(S\) is bounded by \(\Delta\), then \((s, t)\) is \(S\)-normal iff \(\partial s\) and \(t\) have no nontrivial common left-divisor.
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the element \(s'\) s.t. \(ss' = \Delta\)
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- **Lemma (Bessis–Digne–Michel):** The element \(a_{i,j}\) left- (or right-) divides \(a_\lambda\) iff the chord \((i, j)\) is included in the polygon of \(\lambda\).
The normality relation

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![Diagram](image-url)
• When is \((a_\lambda, a_\mu) \cdot S^*_n\)-normal?

• Recall: If a Garside structure \(S\) is bounded by \(\Delta\), then \((s, t)\) is \(S\)-normal iff \(\partial s\) and \(t\) have no nontrivial common left-divisor.
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• **Proposition (Biane).** — The generating series $G(z)$ of $\mathcal{N}_{n,2}^{\text{BKL}^+}$ is derived from the generating series $F(z)$ of $\text{Cat}_n^2$ by

$$G(z) = F(zG(z)).$$

(♯)
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• Proof:
  
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• Proof:
  ▶ Let $G(z) = \sum_n N_{n,2}^{BKL^+} z^n$,
  
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• **Proof:**
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  - From what we saw: $(M_n^\ast)_{\lambda,\mu} = 1$ iff $\overline{\lambda} \land \mu = 0_n$. 
• **Proposition (Biane).**— The generating series $G(z)$ of $N_{n,2}^{BKL+}$ is derived from the generating series $F(z)$ of $\text{Cat}_n^2$ by
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  ▶ From what we saw: $(M_n^*)_{\lambda,\mu} = 1$ iff $\overline{\lambda} \wedge \mu = 0_n$. As $\lambda \rightarrow \overline{\lambda}$ is a bijection, one has also $N_{n,2}^{BKL+} = \#\{(\lambda, \mu) \in (\text{NC}_n)^2 \mid \lambda \lor \mu = 1_n\}$. 
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where $X_1, X_2$ are independent free random variables of variance 1.

Hence connected to the g.f. $F$ of pairs of noncrossing partitions under (\#). \qed
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  ▶ most results involving braids extend to Artin–Tits groups of spherical type (i.e., associated with a finite Coxeter group);
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  ▶ also: exotic Garside structures on braid groups;
  ▶ and exotic non-Garside normal forms with local characterizations;
  ▶ most results involving braids extend to Artin–Tits groups of spherical type (i.e., associated with a finite Coxeter group);
  ▶ many potential combinatorial problems
• Whenever a group admits a finite Garside structure,
  there is a finite state automaton, whence an incidence matrix.

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• **P. Dehornoy**, Combinatorics of normal sequences of braids,  

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  EMS Tracts in Mathematics (2015), www.math.unicaen.fr/~garside/
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