Isolated orderings on an orderable group

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• A simple scheme for constructing monoids in which left-divisibility is a linear ordering, connected with non-Noetherian Garside theory.

• Application: ordered groups whose space of orderings has an isolated point.
Plan:

1. The space of orderings of an orderable group
2. Right-triangular presentations
3. The case of braid groups
1. The space of orderings of an orderable group
2. Right-triangular presentations
3. The case of braid groups
• **Definition.**— A group $G$ is **orderable** if there exists a linear ordering $\leq$ on $G$ that is left-invariant, that is, $g \leq h$ implies $fg \leq fh$ for all $f, g, h$.

• **Lemma.**— (i) If $\leq$ is a left-invariant ordering on $G$, then $P := \{g \in G \mid g > 1\}$ is a subsemigroup of $G$ s.t. $P, P^{-1}, \{1\}$ partition $G$. $P$ : the positive cone of $\leq$. (ii) Conversely, if $P$ is a subsemigroup of $G$ s.t. $P, P^{-1}, \{1\}$ partition $G$, then $g^{-1}h \in P$ defines a left-invariant linear ordering on $G$.

• **Definition.**— A monoid $M$ is of **right-$O$-type** if $M$ is left-cancellative, has no nontrivial invertible element, and the left-divisibility relation $\preceq$ is a linear ordering on $M$.

$$g \preceq h \iff \exists h' (gh' = h)$$

• **Lemma.**— (i) If $\leq$ is a left-invariant ordering on $G$, then $\{g \in G \mid g > 1\}$ is a monoid of $O$-type. (ii) Conversely, if $M$ is a monoid of $O$-type, then $g^{-1}h \in M \setminus \{1\}$ defines a left-invariant linear ordering on the enveloping group of $M$.

_constructing orderable groups $\iff$ constructing monoids of $O$-type_
**Definition.**— For $G$ orderable group, 

$$LO(G) := \text{the family of all positive cones of left-invariant orderings on } G.$$ 

a subset of $\mathcal{P}(G)$  

subsets of $G$  

$\approx \{0, 1\}^G$, a compact totally disconnected space

**Proposition (Sikora).**— The set $LO(G)$ is a closed subspace of $\{0, 1\}^G \times G$.

**Proof:**
- $P$ belongs to $LO(G)$ iff
  
  $$P^2 \subseteq P, \text{ and } P \cup P^{-1} \cup \{1\} = G \text{ and } P \cap P^{-1} = \emptyset \text{ and } 1 \notin P.$$ 

- $P$ does not belong to $LO(G)$ iff $\exists g, h \ (g \in P \& h \in P \& gh \notin P)$ or...

- base of open sets 

$$U_{g_1, \ldots, g_p h_1, \ldots, h_q} = \{X \subseteq G \mid g_1, \ldots, g_p \in X \& h_1, \ldots, h_q \notin X\}.$$  

□
• If $G$ is (finite or) countable, then $\mathcal{P}(G)$ is metrizable.

• Proposition (Linnel).— A space $LO(G)$ cannot be countably infinite.

• Corollary.— If $G$ is countable and orderable, the space $LO(G)$ is
  - either finite,
  - or isomorphic to the Cantor space,
  - or isomorphic to a subspace of the Cantor space with isolated points.

• Examples:
  - $LO(\pi_1(\text{Klein bottle})) \ (= LO(\mathbb{Z} \times \mathbb{Z})))$ has 4 elements;
  - (Sikora) $LO(\mathbb{Z}^n)$ is a Cantor space;
  - (McCleary, Navas) $LO(F_n)$ is a Cantor space.

⇝ Can $LO(G)$ be infinite with isolated points?
• Lemma.— (i) A left-invariant ordering $\leq$ of $G$ is isolated iff exists a finite subset $\{g_1, \ldots, g_p\}$ of $G$ s.t. $\leq$ is the only left-invariant ordering with $1 < g_1, \ldots, 1 < g_p$.
   (ii) This is true in particular if the positive cone is finitely generated as a semigroup.

• Proof: (i) $\{P_{\leq}\} = U_{g_1, \ldots, g_p, \emptyset}$; (ii) if $P_{\leq}$ is generated by $g_1, \ldots, g_p,$
   then every cone that contains $g_1, \ldots, g_p$ includes $P_{\leq}$, hence is equal to $P_{\leq}$. □

• Proposition.— Assume that the group $G$ admits a positive presentation $\langle S \mid R \rangle$ with
   $S$ finite and $\langle S \mid R \rangle^+$ of $O$-type. Then the subsemigroup of $G$ generated by $S$ is the
   positive cone of an isolated left-invariant ordering of $G$.

• Example: $\mathbb{Z} \rtimes \mathbb{Z}$
   $\quad = \langle a, b \mid ab = b^{-1}a \rangle$
   $\quad = \langle a, b \mid a = bab \rangle$. 
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Triangular presentations

• Goal: Constructing finitely generated monoids of $O$-type.
  
  Here: consider presentations of a certain simple syntactic type.

• **Definition.**— A (positive) presentation is **right-triangular** if there exists an enumeration $(s_1, s_2, \ldots)$ of $S$ such that $R$ consists of relations $s_1 = s_2 w_2$, $s_2 = s_3 w_3$, ... $(w_2, w_3, \ldots$ words in $S)$.

• Example: $\langle a, b, c \mid a = bac, b = cba \rangle$ is right-triangular and **left-triangular**.

• **Key Lemma.**— If $(S, R)$ is right-triangular, then TFAE
  (i) $\langle S \mid R \rangle^+$ is of right-$O$-type;
  (ii) any two elements of $\langle S \mid R \rangle^+$ admit a common right-multiple.

• Proof: Right-reversing is necessarily complete; it necessarily provides a $\preceq$-relation. □

  How to prove the existence of common right-multiples?
To prove that common right-multiples exist: find a (right-pre)-Garside element.

Lemma.— Assume that $M$ is a left-cancellative monoid and exists $\Delta$ in $M$ s.t.
(i) Every right-divisor of $\Delta$ is a left-divisor of $\Delta$,
(ii) The left-divisors of $\Delta$ generate $M$.
Then any two elements of $M$ admit a common right-multiple.

Proof: Every element of $M$ left-divides $\Delta^p$ for $p$ large enough.

Proposition.— Assume that $M$ is a monoid that admits a right-triangular presenta-
tion $\langle S \mid R^+ \rangle$ and there exists $\Delta$ in $M$ satisfying $s \prec \Delta \preccurlyeq s\Delta$ for every $s$ in $S$.
Then $M$ is of right-$O$-type (and $\Delta$ is a right-Garside element in $M$).

Proof: Construct an endomorphism $\phi$ of $M$ s.t. $g\Delta = \Delta\phi(g)$ for every $g$.

An easy criterion, in particular well-fitted for computer experiments.
Monoids of $O$-type: examples

• Proposition.— Let $M_{p,q,r} := \langle a, b \mid a = b(a^p b^r)^q \rangle^+$ with $\Delta = a^{p+1}$. Then $M_{p,q,r}$ is of right-$O$-type; for $r = 1$, it is of $O$-type, (and $\Delta$ is a Garside element).

• Proof: Relations $b \preceq a \preceq \Delta \preceq a\Delta$ straightforward; remains to check $\Delta \preceq b\Delta$.
Previous proposition $\Rightarrow$ right-$O$-type; for $r = 1$, everything is symmetric. $\square$

• Particular cases:
  - $M_{1,1,1}$ $a = bab$: Klein bottle group;
  - $M_{1,2,1}$ $a = ba^2b$: braid group $B_3$ with $a = \sigma_1 \sigma_2$, $b = \sigma_2^{-1}$,
    $\leadsto$ hence $LO(B_3)$ has an isolated point;
  - $M_{1,3,1}$ $a = ba^3b$: braid group $B_3$ with $a = \sigma_1 \sigma_2 \sigma_1$, $b = \sigma_2^{-1}$;
  - $M_{p,q,1}$ $x^{p+1} = y^{q+1}$ torus knot group.
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• The D-ordering on $B_n$: a braid is larger than 1 if it admits an expression in the generators $\sigma_i$ s.t. the generator with least index occurs positively only.

**Proposition.** (Navas) The D-ordering is the limit of its conjugates.

$\Rightarrow$ hence not isolated in the space $LO(B_n)$

• **Proposition.** (Dubrovina-Dubrovin) The submonoid $B_n^{\oplus}$ of $B_n$ generated by $\sigma_1 \sigma_2 \cdots \sigma_{n-1}$, $(\sigma_2 \cdots \sigma_{n-1})^{-1}$, $\sigma_3 \cdots \sigma_{n-1}$, $(\sigma_4 \cdots \sigma_{n-1})^{-1}$, ... is of O-type.

$\Rightarrow$ hence isolated in the space $LO(B_n)$

• The monoid $B_3^{\oplus}$ admits the presentations $\langle a, b \mid a = ba^2b \rangle^+$ and $\langle a, b \mid ba^3b \rangle^+$.

$\Rightarrow$ = the monoids of O-type obtained above
• **Proposition.**— The monoid $B_4^\oplus$ admits no right-triangular presentation with respect to the generators $\sigma_1 \sigma_2 \sigma_3, (\sigma_1 \sigma_2)^{-1}, \sigma_3$.

many orderings escape to the current approach

• **Definition.**— An element $\Delta$ of a cancellative monoid $M$ is a Garside element in $M$ if
  - the left- and right-divisors of $\Delta$ coincide,
  - the divisors of $\Delta$ generate $M$,
  - for every $g$ in $M$, the elements $g$ and $\Delta$ admit a left-gcd.

• **Proposition.**— Every submonoid of $O$-type of $B_n$ admits $\Delta_{n}^{\pm 2}$ as a Garside element.

• Proof: The generators $\sigma_i$ are pairwise conjugated under roots of $\Delta_{n}^{2p}$.

  many exotic (non-Noetherian) Garside structures on $B_n$. 
References

For isolated orderings:


For monoids of $O$-type and right-triangular presentations:

- P. Dehornoy; *Monoids of $O$-type, subword reversing, and ordered groups*; arXiv:1204.3211

For orderings on the braid groups:


For non-Noetherian Garside structures: