

Set Theory fifty years after Cohen

Abstract

- Cohen's work is **not** the end of History.
- Today **much** more is known about sets and infinities.
- There is a reasonable hope that the **Continuum Problem** will be **solved**.

Plan

- 1873–1963: The Continuum Problem up to Cohen
- 1963–1987: The first step in the post-Cohen theory
- 1987–present: Toward a solution of the Continuum Problem

1. 1873–1963: The Continuum Problem up to Cohen

- **Theorem (Cantor, 1873).**— There exist at least two non-equivalent infinities.

- **Theorem (Cantor, 1880's).**— There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence

$$\aleph_0 < \aleph_1 < \aleph_2 < \dots < \aleph_\omega < \dots$$



- **Facts.** - $\text{card}(\mathbb{N}) = \aleph_0$,
- $\text{card}(\mathbb{R}) = \text{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} > \text{card}(\mathbb{N})$.

- **Question (Continuum Problem).**— For which α does $\text{card}(\mathbb{R}) = \aleph_\alpha$ hold?

- **Conjecture (Continuum Hypothesis, Cantor, 1879).**— $\text{card}(\mathbb{R}) = \aleph_1$.

↔ equivalently: Every uncountable set of reals has the cardinality of \mathbb{R} .

- **Theorem (Cantor–Bendixson, 1883).**— Closed sets satisfy CH.

Every uncountable closed set of reals has the cardinality of \mathbb{R} .

- **Theorem (Alexandroff, 1916).**— Borel sets satisfy CH.

... and then no progress for 70 years.

- In the meanwhile: Formalization of First Order logic (**Frege, Russell, ...**) and axiomatization of Set Theory (**Zermelo, then Fraenkel, ZF**)

↪ **Consensus:** “We agree that these properties express our current intuition of sets (but this may change in the future)”.

- **First question.**— Is CH or \neg CH (negation of CH) **provable** from ZF?



- **Theorem (Gödel, 1938).**— Unless ZF is contradictory, \neg CH cannot be proved from ZF.

- **Theorem (Cohen, 1963).**— Unless ZF is contradictory, CH cannot be proved from ZF.

- **Conclusion.**— The system ZF is **incomplete**.
 \rightsquigarrow Discover further properties of sets, and adopt an extended list of axioms!

- **Question.**— How to recognize that an axiom is **true**? (?)

Example: CH **may** be taken as an additional axiom, but **not** a good idea...

2. 1963–1987: The first step in the post-Cohen theory

- **Definition.**— For $A \subseteq \mathbb{R}$, consider the two player $\{0, 1\}$ -game G_A :

I	a_1	a_3	...	
II		a_2	a_4	...

where I wins if the real $[0, a_1 a_2 \dots]_2$ belongs to A . Then A is called **determined** if one of the players has a winning strategy in G_A .

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \dots ([0, a_1 a_2 \dots]_2 \in A) \text{ or } \forall a_1 \exists a_2 \forall a_3 \dots ([0, a_1 a_2 \dots]_2 \notin A).$$

- A model for many properties: there exist coding maps $C_{\mathcal{L}}, C_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \rightarrow \mathfrak{P}(\mathbb{R})$ s.t.
 - A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
 - A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always **true** for simple sets:

All closed sets are determined (**Gale–Stewart**, 1962),

All Borel sets are determined (**Martin**, 1975).

- Always **false** for complicated sets:

“All sets are determined” contradicts **AC** (**Mycielski–Steinhaus**, 1962),

“All **projective sets** are determined” unprovable from **ZF** (\approx **Gödel**, 1938).

↑
closure of Borel sets under continuous image and complement

- **Definition.**— The Axiom of **Projective Determinacy (PD)** is the statement “Every projective set of reals is determined”.

- **Propositions (Moschovakis, Kechris, ..., 1970's).**— When added to **ZF**, **PD** provides a complete and satisfactory description of projective sets of reals.

↑
heuristically complete

↑
no pathologies: Lebesgue measurable, etc.

- **Example.**— Under **ZF+PD**, projective sets satisfy **CH**.
- So **PD** is a **useful** axiom, **but not** a natural one (why consider this axiom?),
contrary to large cardinal axioms, which are natural but (a priori) not useful.

3. 1987–present: Toward a solution of the Continuum Problem

- (\approx Cohen) CH and \neg CH not provable from ZF+PD.
 - \rightsquigarrow Adding PD to ZF is only the first (second) step.
- So far three approaches (with three theorems of Woodin):
 - Neutralizing forcing: "generic absoluteness" (1990's)
test-approach, but limited
 - Restricting to forcing-invariant properties: "generic multiverse" (\dagger 2005)
a dead end
 - Identifying one satisfactory universe: "ultimate-L" (1938-2006-present)
currently most promising

Approach 1: **Neutralizing** forcing

- **Cohen's** method of **generic extensions**: analogous to algebraic extensions
for K a field, a larger field $K[\alpha]$ controlled from within K ;
for M a **universe**, a larger universe $M[G]$ **controlled from within** M .

any structure that [↑]satisfies the axioms of ZF “**forcing**”[↑]

- Example (**Cohen**, 1963): From M satisfying **CH**, extension $M[G]$ satisfying \neg **CH**.
- Many properties can be changed using forcing,
but **not** the properties of (hereditarily) **finite** sets : cannot change $2 + 2 = 4$.

- **Theorem** (folklore, 1960's).— Under **ZF**,
properties of hereditarily finite sets are **generically absolute**.

[↑]
invariant under forcing

(explains why ZF is heuristically complete as for properties of hereditarily finite sets)

- **Theorem** (**Foreman–Magidor–Shelah**, 1988).— Under **ZF+PD**,
properties of hereditarily countable sets are generically absolute.

Approach 2: **Restricting** to forcing-invariant properties

- Possible (approach) viewpoint: There is no way to prefer one universe or another one, in particular a generic extension.
- Hence introduce the **generic multiverse**,
↑
 smallest family of universes that is closed under generic extension
 and consider as **valid** only those properties that are satisfied in **all** universes of the generic multiverse (and consider the others, e.g., CH, as meaningless).

• **Theorem (Woodin, 2005).**— Under $ZF+PD$, if the Ω -conjecture is true, the family of all statements that are valid in the sense above has the same algorithmic complexity as the family of all true statements of third-order arithmetic.

involving \mathbb{N} , $\mathcal{P}(\mathbb{N})$, and $\mathcal{P}(\mathcal{P}(\mathbb{N}))$

Turing reducibility

- But the complexity of larger and larger fragments **should** be higher and higher.
↪ **Impossible** to stick to such a point of view...

Approach 3: **Identifying** one satisfactory universe

- As the multiverse approach is impossible, try to identify **one** distinguished universe that could be adopted as a satisfactory reference.
- Typical candidate: Gödel's universe L of **constructible sets** (1938).
the minimal universe (cf. prime subfield): only **definable** sets
- Fully understood: “**fine structure**” theory (**Jensen, Silver**, ..., 1970's)
... but **impossible** as a reference universe:
 - incompatible with large cardinals: does not satisfy **PD**,
 - implies pathologies: existence of a non-measurable projective subset of \mathbb{R} .
- **Question.**— Can one find an **L -like** universe that is compatible with large cardinals?

- (Kunen, 1971) Universe $L[U]$: compatible with large cardinals up to the level of **one measurable cardinal**
- (Mitchell–Steel, 1980–90’s) Universe $L[E]$: compatible with large cardinals up to the level of **PD** (infinitely many Woodin cardinals)
- **But**: how to hope completing the program, as there is an endless hierarchy of increasingly complex large cardinals?

• **Theorem** (Woodin, 2006).— There exists an explicit level (one **supercompact** cardinal) such that the (possible) L -like universe that is compatible with large cardinals up to that level is automatically compatible with all large cardinals.

↑
“ultimate- L ”

- **Now**, a realistic hope to complete the program.
- Still to do (2011): Give an **explicit** construction of **ultimate- L** , and complete the proof that it is **L -like** (= as canonical and well understood as L , $L[U]$, $L[E]$).

- **Conjecture (Woodin, 2010).**— $ZF+PD+V=\text{ultimate-}L$ is true.

an explicit axiom expressing that the universe is (intrinsically) $\text{ultimate-}L$.

↪ proving that the axiom $V=\text{ultimate-}L$ has the same quality as PD ,

(Woodin) true = “validated on the basis of accepted and compelling principles of infinity”.

- **Proposition.**— $ZF+PD+V=\text{ultimate-}L$ implies GCH (and the Ω -conjecture).

↪ If $ZF+PD+V=\text{ultimate-}L$ becomes accepted as the base of Set Theory, then the Continuum Problem will have been solved (at last).

- In any case: possibility of a coherent theory **beyond ZF**
and of a **solution** of the Continuum Problem.

- **W. Hugh Woodin**, *Strong axioms of infinity and the search for V* ,
Proceedings ICM Hyderabad 2010, pp. 504–528