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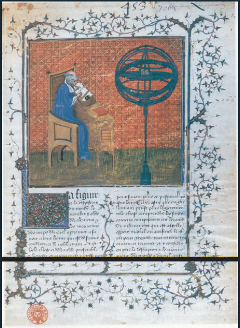

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Patrick Dehornoy

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Nicolas Oresme, Université de Caen

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Plan

- 1873–1963: The Continuum Problem up to Cohen
- 1963–1987: The first step in the post-Cohen theory
- 1987–present: Toward a solution of the Continuum Problem

1. 1873–1963: The Continuum Problem up to Cohen

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Example: CH may be taken as an additional axiom, but not a good idea...

2. 1963–1987: The first step in the post-Cohen theory

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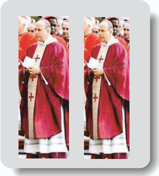

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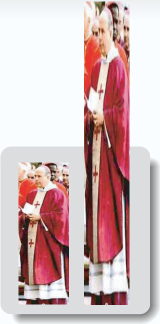

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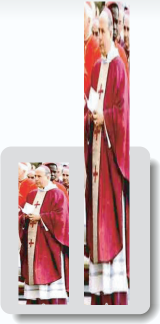

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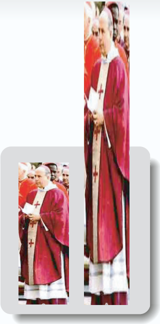

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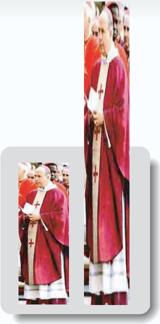

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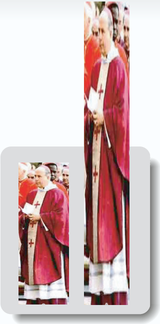

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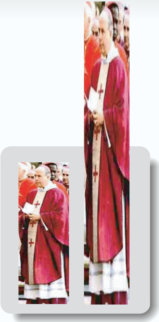

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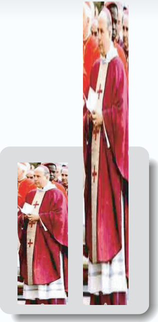

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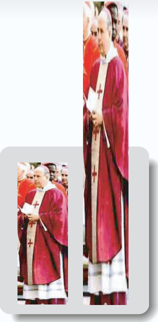

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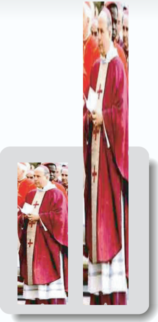

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- $\boldsymbol{X}$ **infinite:** $\exists j : X \to X$ **(j injective not bijective)**
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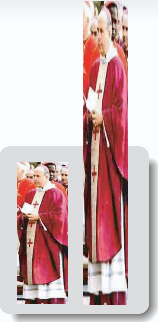

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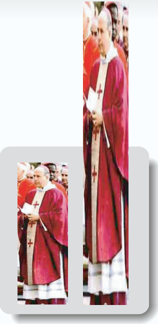

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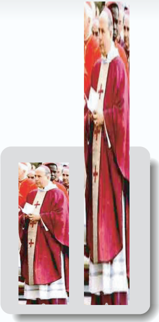

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$$\exists a_1 \forall a_2 \exists a_3 ... ([0, a_1 a_2 ...]_2 \in A) \text{ or } \forall a_1 \exists a_2 \forall a_3 ... ([0, a_1 a_2 ...]_2 \notin A).$$

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- Consensus: The base system for 21th century Set Theory is no longer ZF, but ZF+PD.

3. 1987–present: Toward a solution of the Continuum Problem

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Approach 1: Neutralizing forcing

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- Theorem (Foreman–Magidor–Shelah, 1988).— Under ZF+PD,
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Approach 2: Restricting to forcing-invariant properties

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- But the complexity of larger and larger fragments should be higher and higher.

  $\rightsquigarrow$ Impossible to stick to such a point of view...

Approach 3: Identifying one satisfactory universe

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- **Question.**— Can one find an L-like universe that is compatible with large cardinals?

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- **W. Hugh Woodin**, *Strong axioms of infinity and the search for* V, **Proceedings ICM Hyderabad 2010, pp. 504–528**