A strategy for constructing van Kampen diagrams for semigroups, with an application to the combinatorial distance between the reduced expressions of a permutation.
Plan:

• The general case:
  - Subword reversing as a strategy for constructing van Kampen diagrams
  - Subword reversing as a syntactic transformation
  - A cancellativity criterion

• The case of permutations:
  - bounds for the combinatorial distance between reduced expressions of a permutation
  - recognizing the optimality of a van Kampen diagram
all relations of the form $u = v$ with $u, v$ nonempty words on $S$

• Let $(S, R)$ be a semigroup presentation. Then two words $w, w'$ on $S$ represent the same element of the monoid $\langle S \mid R \rangle^+$ if and only if there exists an $R$-derivation from $w$ to $w'$.

• Proposition (van Kampen,?): If $(S, R)$ is a semigroup presentation, two words $w, w'$ on $S$ represent the same element of the monoid $\langle S \mid R \rangle^+$ if and only if there exists a van Kampen diagram for $(w, w')$.

a tesselated disk with (oriented) edges labeled by elements of $S$ and faces labelled by relations of $R$, with boundary paths labelled $w$ and $w'$. 
• Example: Let $B_n^+ = \left\langle s_1, \ldots, s_{n-1} \mid s_is_j = s_js_i \text{ for } |i-j| = 1 \text{ for } |i-j| \geq 2 \right\rangle$

$(the \ n\text{-strand Artin braid monoid)}.$

Then

![Diagram]

is a van Kampen diagram for $(s_1s_2s_1s_3s_2s_1, s_3s_2s_3s_1s_2s_3).$
• How to build a van Kampen diagram (when it exists)?

\[ \cong \text{solve the word problem: decide } w \equiv^+_R w' \]

• **Subword reversing** = the left strategy: starting with two words \( w, w' \),

- look at the leftmost pending pattern

- choose a relation \( sv = tu \) of \( R \) to close it into \( svu \), and repeat.

**Facts:**
- May not be possible (no relation \( s... = t... \));
- May not be unique (several relations \( s... = t... \));
- May never terminate (when \( u, v \) have length more than 1);
- May terminate but boundary words are longer than \( w, w' \) (certainly happens if \( w, w' \) are not \( R \)-equivalent).
The subword reversing strategy

- At least: deterministic whenever $R$ is a complemented presentation: for each pair of letters $s, t$ in $S$, there is exactly one relation $s... = t...$ in $R$.

- Example: Let $B^+_n = \langle s_1, \ldots, s_{n-1} \mid s_is_j s_i = s_j s_i s_j \text{ for } |i - j| = 1 \text{ and } s_i s_j = s_j s_i \text{ for } |i - j| \geq 2 \rangle^+$. Applying the reversing strategy to $s_1 s_2 s_1 s_3 s_2 s_1$ and $s_3 s_2 s_3 s_1 s_2 s_3$:

So, on this particular example, the reversing strategy works.
• Another way of drawing the same diagram:

only vertical and horizontal edges,
plus dotted arcs connecting vertices that are to be identified
in order to get an actual van Kampen diagram.
• In this way, a uniform pattern:

\[
\begin{array}{c}
s \downarrow \quad t \\
\end{array}
\quad \text{becomes} \quad
\begin{array}{c}
s \downarrow \\
\end{array}
\begin{array}{c}
t \quad u \\
v \quad \text{for } sv = tu \text{ in } R
\end{array}
\]

• More exactly:

\[
\begin{array}{c}
s \downarrow \quad t \\
\end{array}
\quad \text{becomes} \quad
\begin{array}{c}
s \downarrow \\
\end{array}
\begin{array}{c}
t \quad u \\
v \quad \text{for } sv = tu \text{ in } R
\end{array}
\]

including

\[
\begin{array}{c}
s \downarrow \quad s \\
\end{array}
\quad \text{becomes} \quad
\begin{array}{c}
s \downarrow \\
\end{array}
\begin{array}{c}
s \\
\end{array}
\]
• Introduce two types of letters:
  - $S$ for horizontal edges, $S^{-1}$ for vertical edges;
  - read words the Mull of Kintyre to the Pentland Fifth (SW to NE).

• Basic step:

  \[
  \begin{align*}
  &s \implies stu^{-1}v^{-1} \\
  \text{including} & \\
  &s \implies stu^{-1}v^{-1},
  \end{align*}
  \]

  \[
  \begin{align*}
  &s \implies s \implies \varepsilon \\
  \text{the empty word} & \end{align*}
  \]

• Syntactically, “subword reversing”: replacing $-+$ with $+-$. 
• **Definition:** For \(w, w'\) words on \(S \cup S^{-1}\), declare \(w \rightleftharpoons_{R}^{(1)} w'\) if
\[
\exists s, t, u, v \ (sv = tu \text{ lies in } R \text{ and } w = \ldots s^{-1}t\ldots \text{ and } w' = \ldots vu^{-1}\ldots).
\]
Declare \(w \rightleftharpoons_{R} w'\) if there exist \(w_0, \ldots, w_p\) s.t.
\[
w_0 = w, \ w_p = w', \text{ and } w_i \rightleftharpoons_{R}^{(1)} w_{i+1} \text{ for each } i.
\]

• Terminal words: \(w'w^{-1}\) with \(w, w'\) words on \(S\) (no letter \(s^{-1}\)).

• **Lemma:** If \(w, w', v, v'\) are words on \(S\) and \(w^{-1}w' \rightleftharpoons_{R} v'v^{-1}\),
\[
\text{i.e., } w \rightleftharpoons_{R} v, \text{ then } wv' \equiv_{R}^{+} w'v.
\]

• In particular, if \(w^{-1}w' \rightleftharpoons_{R} \varepsilon\), i.e., if \(w \rightleftharpoons_{R} \varepsilon\), then \(w \equiv_{R}^{+} w'\).

↑ the empty word
• Conversely, does $w \equiv_{R}^{+} w'$ implies $w^{-1}w' \rightsquigarrow_{R} \varepsilon$?

• **Definition:** A presentation $(S, R)$ is called **complete** (w.r.t. subword reversing) if $w \equiv_{R}^{+} w'$ implies $w^{-1}w' \rightsquigarrow_{R} \varepsilon$.

hence ... is equivalent to ...

• **Remark:** Completeness implies the solvability of the word problem only if one knows that reversing always terminates.

• **Two questions:**
  - How to recognize completeness?
  - What to do with a complete presentation?
Theorem: (D., '97) Assume that \((S, R)\) is a **homogeneous** complemented presentation. Then \((S, R)\) is complete if, and only if, for each triple \(r, s, t\) in \(S\), the **cube condition** for \(r, s, t\) is satisfied.

- **homogeneous**: \(\exists \text{ } R\)-invariant \(\lambda : S^* \rightarrow \mathbb{N} \ (\lambda(sw) > \lambda(w))\).

- **cube condition** for a triple of positive words \(u, v, w\):

  ...hence checkable (for one triple)
• **Proposition:** Assume that \((S, R)\) is a complete complemented presentation. Then the monoid \(\langle S \mid R \rangle^+\) is left-cancellative.

\[
sa = sa' \implies a = a'
\]

• **Proof:** Assume \(sw \equiv_R^+ sw'\). Want to prove \(w \equiv_R^+ w'\).
Completeness implies: \((sw)^{-1}(sw') \bowtie_R \varepsilon\), i.e., \(w^{-1}s^{-1}sw' \bowtie_R \varepsilon\).

The first step must be \(w^{-1}s^{-1}sw' \bowtie_R w^{-1}w'\), so the sequel must be \(w^{-1}w' \bowtie_R \varepsilon\), hence \(w \equiv_R^+ w'\).
Proposition: Assume that \((S, R)\) is a complete complemented presentation and there exists a finite set \(\hat{S}\) including \(S\) and \textbf{closed under reversing}. Then the word problem of \(\langle S \mid R \rangle^+\) is solvable in quadratic time, and so is that of \(\langle S \mid R \rangle\) if \(\langle S \mid R \rangle^+\) is right-cancellative.

\[
\forall w, w' \in \hat{S} \ \exists v, v' \in \hat{S} \ (w^{-1}w' \leadsto_R v'v^{-1})
\]

Proof: Reversing terminates in quadratic time: construct an \(\hat{S}\)-labeled grid:

- For \(w, w'\) words on \(S\):
  \(w \equiv_R^+ w'\) iff \(w^{-1}w' \leadsto_R \varepsilon\).

- For \(w\) a word on \(S \cup S^{-1}\):
  assume \(w \leadsto_R v'v^{-1}\);
  then \(w \equiv_R \varepsilon\) iff \(v \equiv_R v'\) iff \(v \equiv_R^+ v'\)
  iff \(v^{-1}v' \leadsto_R \varepsilon\) (double reversing).
Subword reversing as a tool

Range

- For semigroups: in principle, all are eligible: completion procedure (when the cube condition fails).

- For groups: unknown; at least: classical and dual presentations of (generalized) braid groups (and all Garside groups) — but certainly more.

Uses

- Cancellativity criterion;
- Existence of least common multiples, identification of Garside structures;
- Computation of the greedy normal form;
- (with Y. Lafont) Construction of explicit resolutions (whence homology);
- (with B. Wiest) Solution to the word problem (complexity issues);
- (with M. Autord) Combinatorial distance between the reduced expressions of a permutation.
Reduced expressions of a permutation

- Every permutation of \( \{1, \ldots, n\} \) is a product of transpositions:

\[
\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \mid s_i s_j s_i = s_j s_i s_j \quad \text{for } |i - j| = 1 \\
\quad s_i s_j = s_j s_i \quad \text{for } |i - j| \geq 2, s_1^2 = \ldots = s_{n-1}^2 = 1 \rangle.
\]

\( \downarrow \) of minimal length

- Proposition ("Exchange Lemma"): Any two reduced expressions of a permutation are connected by braid relations (no need of using \( s_i^2 = 1 \)).

- Combinatorial distance: \( d(u, v) = \) minimal number of braid relations needed to transform \( u \) into \( v \).

- Question: Bounds on \( d(u, v) \)? (The standard proof of the Exchange Lemma gives an exponential upper bound.)

- Proposition (folklore ?): There exist positive constants \( C, C' \) s.t.

- \( d(u, v) \leq C n^4 \) holds for every permutation \( f \) of \( \{1, \ldots, n\} \) and all reduced expressions \( u, v \) of \( f \),

- \( d(u, v) \geq C' n^4 \) holds for some permutation \( f \) of \( \{1, \ldots, n\} \) and some reduced expressions \( u, v \) of \( f \).
• Here: lower bounds; more specifically:

• **Aim**: Recognize whether a given Van Kampen diagram or reversing diagram is possibly optimal.

\[ \# \text{ faces} = \text{combinatorial distance between the bounding words} \]

• Associate a **braid diagram** with every (reduced) \( s \)-word and use the **names** (or the colors) of the strands that cross (i.e., use a “position vs. name” duality):

\[ s_1 s_2 s_1 \mapsto s_1 s_2 s_1 \mapsto \cdots \mapsto N(w) \mapsto \{1,2\}\{1,3\}\{2,3\} \mapsto \{1,3\}\{2,3\}\{1,2\} \mapsto N(w) \mapsto a \text{ sequence } N(w) \text{ of pairs of integers in } \{1,\ldots,n\}. \]
Lower bounds

- For $S, S'$ sequences of pairs of integers in $\{1, ..., n\}$:
  - $I_3(S, S') = \# \text{ triples } \{p, q, r\} \text{ s.t. }$\{p, q\}, \{p, r\} and \{q, r\} appear in different orders in $S, S'$.
  - $I_{2,2}(S, S') = \# \text{ pairs of pairs } \{\{p, q\}, \{p', q'\}\} \text{ s.t. }$\{p, q\} and \{p', q'\} appear in different orders in $S, S'$.

**Lemma:** If $w, w'$ are two reduced expressions of some permutation, then
\[ d(w, w') \geq I_3(N(w), N(w')) + I_{2,2}(N(w), N(w')). \]

- **Proof:** Each type I braid relation ("hexagon") contributes at most 1 to $I_3$, each type II braid relation ("square") contributes at most 1 to $I_{2,2}$. □

- **Example:** $w = s_1 s_2 s_1 s_3 s_2 s_1$, $w' = s_3 s_2 s_3 s_1 s_2 s_3$.
  Then $N(w) = (\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\})$, $N(w') = (\{3, 4\}, \{2, 4\}, \{2, 3\}, \{1, 4\}, \{1, 3\}, \{1, 2\})$.
  Hence $d(w, w') \geq 4 + 2 = 6$.

- **Question (Conjecture?):** Is the above inequality an equality?
• Back to van Kampen diagrams with the aim of recognizing optimality.
  \[\# \text{ faces} = \text{combinatorial distance between bounding words}\]

• Having given names to the generators \(s_i\) (\(=\) the edges of the diagram), give names to the faces:

  **type I:**
  \[
  \begin{align*}
  \{p, q\} & \quad \{q, r\} & \quad \{p, q\} & \quad \{p, q, r\} & \quad \{q, r\} \\
  \{p, q\} & \quad \{q, r\} & \quad \{p, q\} & \quad \{p, q\} & \quad \{q, r\}
  \end{align*}
  \]

  **type II:**
  \[
  \begin{align*}
  \{p, q\} & \quad \{p', q'\} & \quad \{p, q\} & \quad \{p, q\} & \quad \{p', q'\} \\
  \{p, q\} & \quad \{p, q\} & \quad \{p, q\} & \quad \{p, q\} & \quad \{p', q'\}
  \end{align*}
  \]

• **Criterion 1:** A van Kampen diagram in which different faces have different names is optimal.
Example:
• Example:
Separatrices

• (Again in a van Kampen diagram) connect the edges with the same name:

\[
\begin{align*}
\text{type I:} & \quad \{p, q\} \quad \{q, r\} \quad \{p, q\} \quad \{q, r\} \\
\text{type II:} & \quad \{p', q'\} \quad \{p, q\} \quad \{p', q'\}
\end{align*}
\]

\[\Rightarrow\] for each pair \(\{p, q\}\), an (oriented) curve that connect all \(\{p, q\}\)-edges: the \(\{p, q\}\)-separatrix \(\Sigma_{p, q}\).

\[
\begin{align*}
\text{type I:} & \quad \Sigma_{p, q} \quad \Sigma_{q, r} \\
\text{type II:} & \quad \Sigma_{p, q} \quad \Sigma_{p', q'}
\end{align*}
\]
Example:
• Example:
• **Criterion 2:** A van Kampen diagram in which any two separatrices cross at most once is optimal.

• Question: Is the condition necessary, i.e., do any two separatrices cross at most once in an optimal van Kampen diagram?

• Remark: Compare with “a s-word is reduced iff any two strands in the associated braid diagram cross at most one”.
Separatrices and reversing

- Applies in particular to reversing diagrams
  (viewed as particular van Kampen diagrams):
How are separatrices in a reversing diagram? **Three** types of faces:

- **Type I:**
  \[
  \Sigma \quad \Sigma' \quad \Sigma''
  \]

- **Type II:**
  \[
  \Sigma \quad \Sigma' \quad \Sigma'
  \]

- **Type III:**
  \[
  \Sigma \quad \Sigma
  \]

**Criterion 3:** A reversing diagram containing no type III face is optimal.

**Proof:** In order that two separatrices cross twice, one has to go from horizontal to vertical.
A lower bound result

- An improvement: Same argument when reversing steps are grouped:

  \[
  \text{replace } s_i \xrightarrow{s_j} s_i \xrightarrow{s_j} s_i \quad \text{with} \quad s_i \xrightarrow{s_j} s_j \quad \text{for } |i - j| = 1,
  \]

  corresponding to \[\text{and} \]

- An application:

  - **Proposition**: For each \(\ell\), there exist length \(\ell\) reduced \(s\)-words \(w, w'\) satisfying \(w^{-1}w' \xrightarrow{R} v'v^{-1}\) and \(d(wv', w'v) \geq \ell^4 / 8\).

By contrast: for fixed \(n\), Garside’s theory gives an upper bound in \(O(\ell^2)\).
Two conclusions:

- Even in the simple (?) case of braids and permutations, many open questions.
- Importance of having van Kampen diagrams included in a grid.
• P. Dehornoy, Deux propriétés des groupes de tresses  

• F.A. Garside, The braid group and other groups  

• K. Tatsuoka, An isoperimetric inequality for Artin groups of finite type  

• R. Corran, A normal form for a class of monoids including the singular braid monoids  

• P. Dehornoy, Complete positive group presentations;  

• P. Dehornoy & Y. Lafont, Homology of Gaussian groups  

• P. Dehornoy & B. Wiest, On word reversing in braid groups  

• P. Dehornoy & M. Autord, On the combinatorial distance between expressions of a permutation  
  in preparation.

www.math.unicaen.fr/~dehornoy