Unprovability results involving braids
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Patrick Dehornoy
Laboratoire de Mathématiques Nicolas Oresme
Université de Caen
• **Aim**: Describe combinatorial statements involving braids that are unprovable in weak subsystems of Peano arithmetic.
joint work with A. Weiermann, L. Carlucci, A. Bovykin

- **Aim**: Describe combinatorial statements involving braids that are unprovable in weak subsystems of Peano arithmetic *contrary to* all usual algebraic and combinatorial properties.
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• **Interest:** - Involves *mainstream* objects and (hopefully) *natural* properties;
joint work with A.Weiermann, L.Carlucci, A.Bovykin

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  - Involves mainstream objects and (hopefully) natural properties;
  - Leads to new questions and results about braids, in particular: a new normal form.
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- **Plan:**
joint work with A. Weiermann, L. Carlucci, A. Bovykin

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  - 1. Braids and their ordering
joint work with A. Weiermann, L. Carlucci, A. Bovykin

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- **Plan:**
  - 1. Braids and their ordering
  - 2. Long sequences in $B_3$
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  - 3. Phase transition in $B_3$
Aim: Describe combinatorial statements involving braids that are unprovable in weak subsystems of Peano arithmetic contrary to all usual algebraic and combinatorial properties.

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Plan:
- 1. Braids and their ordering
- 2. Long sequences in $B_3$
- 3. Phase transition in $B_3$
- 4. Long sequences in $B_n$
• A 4-strand braid diagram
Braids

- A 4-strand braid diagram
• A 4-strand braid diagram = 2D-projection of a 3D-figure:
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• A 4-strand braid diagram = 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:
Braids

- A 4-strand braid diagram = 2D-projection of a 3D-figure:

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Braids

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Braids

- A 4-strand braid diagram = 2D-projection of a 3D-figure:

- isotopy = move the strands but keep the ends fixed:

  isotopic to
• A 4-strand braid diagram = 2D-projection of a 3D-figure:

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• **A 4-strand braid diagram** = 2D-projection of a 3D-figure:

![Diagram](image)

• **isotopy** = move the strands but keep the ends fixed:

![Diagram](image)
Braids

- **A 4-strand braid diagram** = 2D-projection of a 3D-figure:

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• A 4-strand braid diagram = 2D-projection of a 3D-figure:

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• A 4-strand braid diagram = 2D-projection of a 3D-figure:

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Braids

- A 4-strand braid diagram = 2D-projection of a 3D-figure:

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Braids

• A 4-strand braid diagram = 2D-projection of a 3D-figure:

• isotopy = move the strands but keep the ends fixed:

• a braid := an isotopy class $\rightsquigarrow$ represented by 2D-diagram,
Braids

- A 4-strand braid diagram = 2D-projection of a 3D-figure:

- isotopy = move the strands but keep the ends fixed:

- a braid := an isotopy class represented by 2D-diagram, but different 2D-diagrams may give rise to the same braid.
**Braid groups**

- **Product** of two braids:
• **Product** of two braids:

![Diagram of two braids](image)

Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[ \approx \]

because \( \approx \).

For each \( n \), the group \( B_n \) of \( n \) strand braids (E. Artin, ∼1925).
Braid groups

- **Product** of two braids:

\[ \star \text{ then well-defined (w.r.t. isotopy), associative, admits a unit: } \star \approx \]

Then isotopic to and inverses:

\[ braid \cdot braid = braid \cdot braid \]

For each \( n \), the group \( B_n \) of \( n \)-strand braids (E. Artin, \( \sim 1925 \)).
Braid groups

- **Product** of two braids:

  ![Diagram of braid product]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

  ![Diagram of braid product and unit]

For each \( n \), the group \( B_n \) of \( n \)-strand braids (E. Artin, \( \sim \) 1925).
• **Product** of two braids:

\[
\begin{array}{ccc}
\text{Braid} & \times & \text{Braid} \\
\end{array}
\]

Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[
\begin{array}{ccc}
\text{Braid} & \times & \text{Braid} \\
\end{array}
\]
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:
Braid groups

- **Product** of two braids:

  ![Product of two braids diagram]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

  ![Well-defined, associative, unit diagram]

For each \( n \), the group \( B_n \) of \( n \) strand braids (E. Artin, 1925).
Braid groups

- **Product** of two braids:

  ![Braid Product Diagram]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

  ![Braid Units Diagram]

- and inverses:

  ![Braid Inverses Diagram]
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:
**Braid groups**

- **Product** of two braids:

  ![Diagram of braid product]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

  ![Diagram of braid unit]

and inverses:

![Diagram of braid inverse]
Braid groups

- **Product of two braids:**

  ![Diagram of braid product]

  Then well-defined (w.r.t. isotopy), associative, admits a unit:

  ![Diagram of isotopy and unit]

  and inverses:

  ![Diagram of braid inverses]
• **Product** of two braids:

\[ \star \] Then well-defined (w.r.t. isotopy), associative, admits a unit:

and inverses:

For each \( n \), the group \( B_n \) of \( n \)-strand braids (E. Artin, \( \sim 1925 \)).
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:
• **Product** of two braids:

\[ \begin{array}{c}
\text{braid} \\
\text{braid}
\end{array} \quad \ast \quad \begin{array}{c}
\text{braid} \\
\text{braid}
\end{array} \quad \equiv \quad \begin{array}{c}
\text{braid} \\
\text{braid}
\end{array} \]

Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[ \begin{array}{c}
\text{braid} \\
\text{braid}
\end{array} \quad \ast \quad \begin{array}{c}
\text{braid}
\end{array} \quad \equiv \quad \begin{array}{c}
\text{braid}
\end{array} \quad \approx
\]

and inverses:

\[ \begin{array}{c}
\text{braid}
\end{array} \quad \ast \quad \begin{array}{c}
\text{braid}^{-1}
\end{array} \quad \equiv \quad \begin{array}{c}
\text{braid}
\end{array} \quad \text{isotopic to}
\]
• **Product** of two braids:

\[ \star : = \]

Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[ \star = \approx \]

and inverses:

\[ (\text{braid})^{-1} = \text{braid} \]
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:
Braid groups

- **Product** of two braids:

\[
\begin{array}{c}
\text{braid} \quad \ast \quad \text{braid} \\
\end{array}
\]

Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[
\begin{array}{c}
\text{braid} \quad \ast \quad \text{nothing} \\
\approx \\
\end{array}
\]

and inverses:

\[
\text{braid} \quad \ast \quad \text{braid}^{-1} = \text{nothing}
\]
- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:

  \[ \text{braid} \cdot (\text{braid})^{-1} = \text{isotopic to} \]
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

and inverses:

For each \( n \), the group \( B_n \) of \( n \) strand braids (E. Artin, \( \sim 1925 \)).
Braid groups

- **Product** of two braids:

\[
\begin{array}{c}
\text{braid}
\end{array}
\begin{array}{c} \star \end{array}
\begin{array}{c} \text{braid} \end{array}
\begin{array}{c} = \end{array}
\begin{array}{c} \text{braid} \end{array}
\begin{array}{c} \text{braid} \end{array}
\end{array}
\]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[
\begin{array}{c}
\text{braid}
\end{array}
\begin{array}{c} \star \end{array}
\begin{array}{c} \text{braid} \end{array}
\begin{array}{c} = \end{array}
\begin{array}{c} \text{braid} \end{array}
\begin{array}{c} \text{braid} \end{array}
\end{array}
\]

\[
\begin{array}{c} \approx \end{array}
\]

and inverses:

\[
\begin{array}{c}
\text{braid}
\end{array}
\begin{array}{c} -1 \end{array}
\begin{array}{c} = \end{array}
\begin{array}{c} \text{braid} \end{array}
\begin{array}{c} \text{braid} \end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{iso\text{t}opic \ to}
\end{array}
\]
Braid groups

- **Product** of two braids:

  ![Diagram of braid product](image)

  Then well-defined (w.r.t. isotopy), associative, admits a unit:

  ![Diagram of isotopy](image)

  and inverses:

  ![Diagram of braid inverse](image)
• **Product** of two braids:

\[
\begin{array}{c}
\text{braid} \\
\hline
\text{braid}
\end{array}
\]

Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[
\begin{array}{c}
\text{braid} \\
\hline
\text{braid}
\end{array}
\]

and inverses:

\[
\begin{array}{c}
\text{braid} \\
\hline
\text{braid}
\end{array}
\]

Because

\[
\begin{array}{c}
\text{braid} \\
\hline
\text{braid}
\end{array}
\]

For each \( n \), the group \( B_n \) of \( n \) strand braids (E. Artin, ∼1925).
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:
Braid groups

- **Product** of two braids:

- Then well-defined \((w.r.t. \text{ isotopy})\), associative, admits a unit:

- And inverses:
Braid groups

- **Product of two braids:**

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:

  - isotopic to because
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:


For each $n$, the group $B_n$ of $n$-strand braids (E. Artin, ~1925).
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:

  - Because $\approx$
Braid groups

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:

  because
**Braid groups**

- **Product** of two braids:

- Then well-defined (*w.r.t. isotopy*), associative, admits a unit:

- and inverses:

and isotopic to
- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

and inverses:

because
Braid groups

- **Product** of two braids:

\[ \text{braid} \times \text{braid} = [\text{braid}] \]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[ \text{braid} \times [\text{braid}] = [\text{braid}] \approx \]

and inverses:

\[ [\text{braid}]^{-1} = \text{braid} \text{ because } \]

isotopic to
**Braid groups**

- **Product of two braids:**

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:

  because
Braid groups

- **Product** of two braids:

\[
\begin{array}{c}
\text{braid} \\
\end{array}
\quad \ast \\
\begin{array}{c}
\text{braid}
\end{array}
\quad := \\
\begin{array}{c}
\text{braid}
\end{array}
\]

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

\[
\begin{array}{c}
\text{braid}
\end{array}
\quad \ast \\
\begin{array}{c}
\text{braid}
\end{array}
\quad = \\
\begin{array}{c}
\text{braid}
\end{array}
\]

and inverses:

\[
\begin{array}{c}
\text{braid}
\end{array}
\quad\quad -1
\]

\[
\begin{array}{c}
\text{braid}
\end{array}
\quad = \\
\begin{array}{c}
\text{braid}
\end{array}
\quad \approx \\
\begin{array}{c}
\text{braid}
\end{array}
\quad \text{because}
\]

\[
\begin{array}{c}
\text{braid}
\end{array}
\quad \approx \\
\begin{array}{c}
\text{braid}
\end{array}
\quad .
\]
**Braid groups**

- **Product** of two braids:

- Then well-defined (w.r.t. isotopy), associative, admits a unit:

- and inverses:

- For each $n$, the group $B_n$ of $n$ strand braids ($E$.Artin, $\sim1925$).
Artin presentation of $B_n$

- Artin generators of $B_n$: 

\[
\begin{align*}
\sigma_1 
\sigma_2 
\sigma_1 
\sigma_2 
\sigma_1
\end{align*}
\]

\[
\begin{align*}
\sigma_1 
\sigma_3 
\sigma_3 
\sigma_1
\end{align*}
\]
Artin presentation of $B_n$

- Artin generators of $B_n$: 

\[
\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_3 \sigma_3 \sigma_1
\]
Artin presentation of $B_n$

- Artin generators of $B_n$:

\[ \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_3 \sigma_3 \sigma_1 \]
Artin presentation of $B_n$

- Artin generators of $B_n$:

\[
\begin{align*}
\sigma_1 \sigma_2 \sigma_1 & \approx \sigma_2 \sigma_1 \\
\sigma_1 \sigma_3 \sigma_3 \sigma_1 & \approx \sigma_3 \sigma_1
\end{align*}
\]
Artin presentation of $B_n$:

- Artin generators of $B_n$:

\[
\begin{align*}
\sigma_1 
& \approx \sigma_2 
& \approx \sigma_3 
& \sigma_1 
& \approx 
\end{align*}
\]

\[
\begin{align*}
\sigma_1 
& \approx 
\end{align*}
\]

\[
\sigma_1 
\]
• Artin generators of $B_n$:
Artin presentation of $B_n$

- Artin generators of $B_n$:

\[
\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_3 = \sigma_1 \sigma_2 \sigma_3
\]
Artin presentation of $B_n$

- Artin generators of $B_n$:

\[ \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 = \sigma_1^{-1} \]

Theorem (Artin): The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to:

- $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i - j| = 1$,
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$. 

\[ \approx \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \approx \sigma_1 \sigma_3 \sigma_3 \sigma_1 \]
Artin presentation of $B_n$

- **Artin generators of $B_n$:**

\[
\sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1} = \sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1}
\]

- **Theorem (Artin):** The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$. 
Artin presentation of $B_n$

- Artin generators of $B_n$:

  \[ \sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1} \]

- Theorem (Artin): The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

  \[ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1, \]
Artin presentation of $B_n$

- Artin generators of $B_n$:

\[
\sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1}
\]

- Theorem (Artin): The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

\[
\begin{align*}
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| = 2.
\end{align*}
\]
Artin presentation of $B_n$

- **Artin generators of $B_n$:**

  \[
  \begin{array}{cccc}
  \sigma_1 & \sigma_2 & \sigma_3 & \sigma_1^{-1} \\
  \end{array}
  \]

- **Theorem (Artin):** The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

  \[
  \begin{align*}
  \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
  \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| \geq 2.
  \end{align*}
  \]
Artin presentation of $B_n$

- Artin generators of $B_n$:

\[
\sigma_1 \sigma_2 \sigma_1 \sigma_1 = \sigma_1 \sigma_3 \sigma_3 \sigma_1
\]

- Theorem (Artin): The group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$, subject to

\[
\begin{align*}
\sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| \geq 2.
\end{align*}
\]
• Definition: For \( x, y \) in \( B_\infty \), say that \( x < y \) holds if, among all words representing \( x^{-1}y \), at least one is such that the generator \( \sigma_i \) with highest index appears positively only (\( \sigma_i \) occurs, \( \sigma_i^{-1} \) does not).

• Theorem: (i) The relation \( < \) is a left-invariant total order on \( B_\infty \); (ii) (Laver) The restriction of \( < \) to \( B_+\infty \) is a well-order; (iii) (Burckel) The restriction of \( < \) to \( B_+n \) has length \( \omega_\omega n - 2 \).
• Definition: For $x, y$ in $B_{\infty}$, say that $x < y$ holds if, among all words representing $x^{-1}y$, at least one is such that the generator $\sigma_i$ with highest index appears positively only ($\sigma_i$ occurs, $\sigma_i^{-1}$ does not).

$\Rightarrow$ e.g., $\sigma_2 < \sigma_2 \sigma_1$ holds, because $\sigma_2^{-1} \sigma_1 \sigma_2$
• Definition: For \(x, y\) in \(B_{\infty}\), say that \(x < y\) holds if, among all words representing \(x^{-1}y\), at least one is such that the generator \(\sigma_i\) with highest index appears positively only (\(\sigma_i\) occurs, \(\sigma_i^{-1}\) does not).

\[\sigma_2 < \sigma_2\sigma_1\] holds, because \(\sigma_2^{-1}\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1^{-1}\), and, in the latter word, \(\sigma_2\) appears positively only.
The standard braid order

• Definition: For $x, y$ in $B_\infty$, say that $x < y$ holds if, among all words representing $x^{-1}y$, at least one is such that the generator $\sigma_i$ with highest index appears positively only ($\sigma_i$ occurs, $\sigma_i^{-1}$ does not).

  \[
  \sigma_2 < \sigma_2 \sigma_1 \text{ holds, because } \sigma_2^{-1} \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1^{-1},
  \]
  and, in the latter word, $\sigma_2$ appears positively only.

• Theorem: (i) The relation $<$ is a left-invariant total order on $B_\infty$;
The standard braid order

• Definition: For \( x, y \) in \( B_\infty \), say that \( x < y \) holds if, among all words representing \( x^{-1}y \), at least one is such that the generator \( \sigma_i \) with highest index appears positively only (\( \sigma_i \) occurs, \( \sigma_i^{-1} \) does not).

\[ \sigma_2 < \sigma_2 \sigma_1 \] holds, because \( \sigma_2^{-1} \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1^{-1} \), and, in the latter word, \( \sigma_2 \) appears positively only.

• Theorem: (i) The relation \( < \) is a left-invariant total order on \( B_\infty \); (ii) (Laver) The restriction of \( < \) to \( B^+_\infty \) is a well-order;
The standard braid order

• Definition: For $x, y$ in $B_\infty$, say that $x < y$ holds if, among all words representing $x^{-1}y$, at least one is such that the generator $\sigma_i$ with highest index appears positively only ($\sigma_i$ occurs, $\sigma_i^{-1}$ does not).

$\Rightarrow$ e.g., $\sigma_2 < \sigma_2 \sigma_1$ holds, because $\sigma_2^{-1} \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1^{-1}$, and, in the latter word, $\sigma_2$ appears positively only.

• Theorem: (i) The relation $<$ is a left-invariant total order on $B_\infty$;
(ii) (Laver) The restriction of $<$ to $B^+_{\infty}$ is a well-order;
(iii) (Burckel) The restriction of $<$ to $B^+_n$ has length $\omega^{\omega^{-2}}$. 
• Construct (very) long descending sequences of braids using a simple inductive rule.
• Construct (very) **long** descending sequences of braids using a simple inductive rule.

⇝ Reminiscent of Goodstein’s sequences and Hydra battles: “battle against a malevolent braid”: get rid of all crossings; at step $t$, chop off 1 crossing, but $t$ new crossings reappear in general.
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• Here in the 3 strand version—but exists for each $n$. 

Long sequences of braids
- (Burckel) The alternating normal form of a positive 3-braid:

$$\sigma_{[p]}^{e_p} \cdots \sigma_2^{e_2} \sigma_1^{e_1} \text{ with } e_p \geq 1, \, e_k \geq 2 \text{ for } p > k \geq 3, \, e_2 \geq 1, \, e_1 \geq 0,$$

1 or 2 according to $p \pmod{2}$
• (Burckel) The alternating normal form of a positive 3-braid:

$$\sigma^e_p \ldots \sigma^e_2 \sigma^e_1$$

with $e_p \geq 1$, $e_k \geq 2$ for $p > k \geq 3$, $e_2 \geq 1$, $e_1 \geq 0$,

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• The critical position: smallest (= rightmost) $k$ s.t. $e_k$ does not have the minimal legal value, if it exists, $p$ otherwise.
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\[ \sigma_{[p]}^{e_p} \cdots \sigma_2^{e_2} \sigma_1^{e_1} \text{ with } e_p \geq 1, \ e_k \geq 2 \text{ for } p > k \geq 3, \ e_2 \geq 1, \ e_1 \geq 0, \]

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• The **critical position**: smallest (= rightmost) \( k \) s.t. \( e_k \) does not have the minimal legal value, if it exists, \( p \) otherwise.

\[
\begin{array}{c}
\sigma_2 \\
\sigma_1 \\
\end{array}
\]

\[
\begin{array}{c}
4 \\
3 \\
2 \\
1 \\
\end{array}
\]
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• (Burckel) The alternating normal form of a positive 3-braid:
\[ \sigma_{[p]}^{\varepsilon_p} \ldots \sigma_2^{\varepsilon_2} \sigma_1^{\varepsilon_1} \] with \( \varepsilon_p \geq 1, \varepsilon_k \geq 2 \) for \( p > k \geq 3, \varepsilon_2 \geq 1, \varepsilon_1 \geq 0, \)
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1 or 2 according to \( p \) (mod 2)

The critical position: smallest (= rightmost) \( k \) s.t. \( e_k \) does not have the minimal legal value, if it exists, \( p \) otherwise.
The $G_3$-sequence from a positive 3-braid $b$:
\( G_3 \)-sequences

- The \( G_3 \)-sequence from a positive 3-braid \( b \):
  - Start with the alternating normal form of \( b \);
\(G_3\)-sequences

- The \(G_3\)-sequence from a positive 3-braid \(b\):
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\(G_3\)-sequences

- The \(G_3\)-sequence from a positive 3-braid \(b\):
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• The $G_3$-sequence from a positive 3-braid $b$:

- Start with the alternating normal form of $b$;
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**Example:** $\sigma_2^2 \sigma_1^2$. 

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Diagram: 

- Critical block: remove 1 crossing
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The $G_3$-sequence from a positive 3-braid $b$:

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Example: $\sigma_2^2 \sigma_1^2$, $\sigma_2^2 \sigma_1$, $\sigma_2 \sigma_1$, $\sigma_1$, $\sigma_7$, $\sigma_6$, $\sigma_5$, $\sigma_4$, $\sigma_3$, $\sigma_2$, $\sigma_1$, $1$. 
• The $G_3$-sequence from a positive $3$-braid $b$:
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• Example: $\sigma_2^2\sigma_1^2, \sigma_2^2\sigma_1, \sigma_2^2, \sigma_2\sigma_1^3, \sigma_2\sigma_1^2$.
$G_3$-sequences

- The $G_3$-sequence from a positive 3-braid $b$:
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Example: $\sigma_2^2\sigma_1^2$, $\sigma_2^2\sigma_1$, $\sigma_2^2$, $\sigma_2\sigma_1^3$, $\sigma_2\sigma_1^2$, $\sigma_2\sigma_1$, $\sigma_2\sigma_1$. 
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- Example: $\sigma_2^2\sigma_1^2$, $\sigma_2^2\sigma_1$, $\sigma_2^2$, $\sigma_2\sigma_1^3$, $\sigma_2\sigma_1^2$, $\sigma_2\sigma_1$, $\sigma_2$, $\sigma_1^7$. 
The $\mathcal{G}_3$-sequence from a positive 3-braid $b$:

- Start with the alternating normal form of $b$;
- At step $t$: remove 1 crossing in the critical block;
  add $t$ new crossings in the next block, if it exists;
- The sequence stops when (if) one reaches the braid 1.

![Diagram showing the process](image)

- Example: $\sigma_2\sigma_1^2$, $\sigma_2\sigma_1$, $\sigma_2$, $\sigma_2\sigma_1^3$, $\sigma_2\sigma_1^2$, $\sigma_2\sigma_1$, $\sigma_2$, $\sigma_1^7$, $\sigma_1^6$. 
**$G_3$-sequences**

- The $G_3$-sequence from a positive $3$-braid $b$:
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---

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An unprovability statement

- More examples:

Nevertheless:

- Proposition A: Every $G_3$-sequence is finite.
  Proof: $G_3$-sequences are descending sequences in a well-order. □

But:

- Theorem: Proposition A cannot be proved in $I\Sigma_1$.
 ↑ the subsystem of Peano arithmetic in which induction is restricted to formulas with one $\exists$ quantifier in contrast with the folklore result:

- All usual (algebraic) properties of braids can be proved in $I\Sigma_1$. 
• More examples:
  - starting with $\sigma_1 \sigma_2 \sigma_1$ requires 30 steps;
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• More examples:
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An unprovability statement

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in contrast with the folklore result:

• All usual (algebraic) properties of braids can be proved in $I \Sigma_1$.  
  

An unprovability statement
• Unprovability of the finiteness of $G_3$-sequences in $I\Sigma_1$: 

- assign ordinals to braids, and compare with fundamental sequences and the Hardy hierarchy.

- Definition: For $\sigma$ a $G_3$-braid with normal form $\sigma^{e_p}\varepsilon_1 \ldots \sigma^{e_2}\varepsilon_2 \sigma^{e_1}\varepsilon_1$, put
  \[
  \text{ord}(b) := \omega^{p-1} \cdot e_p + \sum_{p > k \geq 1} \omega^{k-1} \cdot (e_k - e_{\min k})
  \]
  where $e_{\min k} = 2$ for $k \geq 3$, $e_{\min 2} = 1$, and $e_{\min 1} = 0$.

- Lemma: For every $G_3$-braid $b$ and every number $t$:
  \[
  \text{ord}(b^{\{t\}}) = \text{ord}(b^{[t]})
  \]

↑ the braid obtained from $b$ at step $t$.

"Fundamental sequence" of ordinals: $\lambda[x] := \gamma + \omega^{r-1} \cdot x$ for $\lambda = \gamma + \omega^{r}$.
Proof of unprovability

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where $e_k^{\text{min}} = 2$ for $k \geq 3$, $e_2^{\text{min}} = 1$, $e_1^{\text{min}} = 0$. 
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The combinatorial principle $\text{WO}_f$

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- Definition: For $f: \mathbb{N} \rightarrow \mathbb{N}$, let $WO_f$ be the combinatorial principle:

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Phase transition

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• Key point: Fine counting arguments in $B_3$, namely evaluating
  \[
  \text{card}\{b \in B_3 \mid \|b\| \leq \ell \ \& \ b < \Delta^k_3\}.
  \]
• Let $S_{k, \ell} := \{ b \in B_3 \mid \|b\| \leq \ell \ \& \ b < \Delta_3^k \}$. 
• Let $S_{k,\ell} := \{ b \in B_3 \mid \| b \| \leq \ell \ & b < \Delta_3^k \}$.

• Proposition: For $\ell \geq k \geq 1$:

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Braid counting

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• Extension to $n$-braids: Two solutions developed so far:
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**Proposition:** Every braid in $B_n^+$ admits a unique decomposition

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A notion of $G_\infty$-sequence similar to $G_3$-sequence, but involving arbitrary braids instead of $3$-braids.
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\text{ but involving arbitrary braids instead of 3-braids.} \]
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