THE GROUP OF PARENTHESIZED BRAIDS
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• An extended braid group $B$ that includes
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  - Artin's braid group $B_{\infty}$, and
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  - Thompson's group $F$, 
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  - also connected (quotient) with groups introduced
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• Here: Insist on
  - similarity with $B_\infty$, in particular
    possibility of using self-distributivity via diagram colourings,
  - connection with homeomorphisms of $S^2 \setminus$ Cantor.
Def. (R. Thompson, 1965): \( F := \langle x_0, x_1, \ldots ; x_j x_i = x_i x_{j+1} \text{ for } j > i \rangle \).
• Def. (R. Thompson, 1965): $F := \langle x_0, x_1, ... ; x_j x_i = x_i x_{j+1} \text{ for } j > i \rangle$.

$F \cong \{ \text{piecewise linear orientation preserving homeo's of } [0, 1] \text{ with dyadic discontinuities of the derivative and slopes of the form } 2^k \}$.
Thompson's Group $F$

- Def. (R. Thompson, 1965): $F := \langle x_0, x_1, \ldots ; x_j x_i = x_i x_{j+1} \text{ for } j > i \rangle$.
- $F \simeq \{\text{piecewise linear orientation preserving homeo's of } [0, 1] \text{ with dyadic discontinuities of the derivative and slopes of the form } 2^k\}$.

Also represented as

- $x_0$ as a continuous function from $[0, 1]$ to $[0, 1]$.
- $x_1$ as a continuous function from $[0, 1]$ to $[0, 1]$.
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\( F \cong \{ \text{piecewise linear orientation preserving homeo's of } [0, 1] \text{ with dyadic discontinuities of the derivative and slopes of the form } 2^k \} \).

\( x_0 \mapsto x_1 \mapsto \ldots \) also represented as

\( x_0 \mapsto x_1 \mapsto \text{one element of } F = \text{a pair of dyadic decompositions of } [0, 1] \):
• Def. (R. Thompson, 1965): $F := \langle x_0, x_1, \ldots ; x_j x_i = x_i x_{j+1} \text{ for } j > i \rangle$.

$F \simeq \{\text{piecewise linear orientation preserving homeo's of } [0, 1] \text{ with dyadic discontinuities of the derivative and slopes of the form } 2^k \}$.

$\leadsto$ one element of $F$ = a pair of dyadic decompositions of $[0, 1]$

$\leadsto$ also: a pair of finite binary rooted trees.
• Ordinary braid diagrams:
Ordinary braid diagrams:

← initial positions: ● ● ● ●

← final positions: ● ● ● ●
• Ordinary braid diagrams:

\[ \begin{array}{c}
\text{initial positions:} \quad \bullet \bullet \bullet \bullet \\
\text{final positions:} \quad \bullet \bullet \bullet \bullet \\
\end{array} \]

\[ \begin{array}{c}
\xrightarrow{\text{one elementary pattern: crossing } \sigma_i} \\
\begin{array}{c}
1 \quad 2 \\
i \quad i+1
\end{array}
\end{array} \]
- Ordinary braid diagrams:

\[ \begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{initial positions:} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array} \]

\[ \begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{final positions:} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array} \]

\[ \begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{one elementary pattern: crossing } \sigma_i : \quad 1 \quad 2 \quad i \quad i+1 \\
\end{array} \]

- Parenthesised braid diagrams:
• Ordinary braid diagrams:
  ← initial positions: • • • •
  ← final positions: • • • •
  ⇝⇝⇝⇝⇝⇝⇝⇝⇝ one elementary pattern: crossing $\sigma_i$:

• Parenthesized braid diagrams:
  ← initial positions: • • • •
  ← final positions: • • •
• Ordinary braid diagrams:

← initial positions: • • • •

← final positions: • • • •

⇝⇝⇝⇝⇝⇝⇝⇝⇝ one elementary pattern: crossing $\sigma_i$:

\[
1 \quad 2 \quad i \quad i+1
\]

• Parenthesized braid diagrams:

← initial positions: (• •)

← final positions: •(• •)
• Ordinary braid diagrams:

\[
\begin{array}{c}
\rightarrow \\
\begin{array}{c}
\text{initial positions:} \\
\bullet \bullet \bullet \bullet \\
\end{array} \\
\rightarrow \\
\begin{array}{c}
\text{final positions:} \\
\bullet \bullet \bullet \bullet \\
\end{array}
\end{array}
\]

\[\Rightarrow\] one elementary pattern: crossing $\sigma_i : 1 \ 2 \ i \ i+1

• Parenthesized braid diagrams:

\[
\begin{array}{c}
\rightarrow \\
\begin{array}{c}
\text{initial positions:} \\
(\bullet \bullet) \\
\end{array} \\
\rightarrow \\
\begin{array}{c}
\text{final positions:} \\
\bullet (\bullet \bullet) \\
\end{array}
\end{array}
\]

\[\Rightarrow\] two elementary patterns: crossing $\sigma_i : 1 \ 2 \ i \ i+1
• Ordinary braid diagrams:

← initial positions: • • • •
← final positions: • • • •
⇝⇝⇝⇝⇝⇝⇝⇝⇝ one elementary pattern: crossing $\sigma_i$:

1 2 $i$ $i+1$

• Parenthesized braid diagrams:

← initial positions: (• •)
← final positions: •(• •)
⇝⇝⇝⇝⇝⇝⇝⇝⇝ two elementary patterns: crossing $\sigma_i$:

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• Parenthesized braid diagrams:

← initial positions: (• •)
← final positions: •(• •)
⇝⇝⇝⇝⇝⇝⇝⇝⇝ two elementary patterns: crossing $\sigma_i$:

1 2 $i$ $i+1$
More precisely:

\[ \sigma_i \rightsquigarrow 1 \ 2 \ 3 \ 4 \ \cdots \ i \ i+1 \]
More precisely:

\[ \sigma_i \sim \quad 1 \quad 2 \quad \ldots \quad i \quad i+1 \]

everything in \([i, i+1)\) crosses over everything in \([i+1, i+2)\)
More precisely:

$$\sigma_i \sim \begin{array}{c}
1 \quad 2 \quad \cdots \quad i \quad i+1
\end{array}$$

everything in $[i, i+1)$ crosses over everything in $[i+1, i+2)$

$$a_i \sim \begin{array}{c}
1 \quad 2 \quad \cdots \quad i \quad i+1
\end{array}$$
• More precisely:

\[ \sigma_i \sim \cdots \sim \ \text{everything in } [i, i + 1) \text{ crosses over everything in } [i + 1, i + 2) \]

\[ a_i \sim \cdots \sim \ \text{preserved} \]
More precisely:

\[ \sigma_i \sim \quad \text{everything in } [i, i+1) \text{ crosses over everything in } [i+1, i+2) \]

\[ a_i \sim \quad \text{preserved everything in } [i, i+1) \text{ is shrunked by an } \varepsilon \text{ factor} \]
• More precisely:

\[ \sigma_i \sim \quad 1 \quad 2 \quad \cdots \quad i \quad i+1 \]

everything in \([i, i+1)\) crosses over everything in \([i+1, i+2)\)

\[ a_i \sim \quad 1 \quad 2 \quad \cdots \quad i \quad i+1 \]

preserved everything in \([i, i+1)\) is shrunk by a \(\varepsilon\) factor

translated everything in \([i+1, i+2)\) is translated
More precisely:

\[ \sigma_i \sim \cdots \sim \]

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
i \\
i+1 \\
\end{array}
\]

everything in \([i, i+1)\) crosses over everything in \([i+1, i+2)\)

\[ a_i \sim \cdots \sim \]

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
i \\
i+1 \\
\end{array}
\]

depicted as

- preserved
- translated

everything in \([i, i+1)\) is shrunked by a \(\varepsilon\) factor

can be formalized using (finite) trees and dyadic numbers
• To make a group $B_\infty$ out of ordinary diagrams, use completion:

\[
\begin{array}{c}
\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\end{array}
\end{array}
\sim
\begin{array}{c}
\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\end{array}
\end{array}
\]
To make a group $B_\infty$ out of ordinary diagrams, use completion:

\[ \text{\includegraphics[width=0.4\textwidth]{diagram1}} \sim \text{\includegraphics[width=0.1\textwidth]{diagram2}} \sim \text{\includegraphics[width=0.3\textwidth]{diagram3}}, \ i.e., \quad \text{\includegraphics[width=0.1\textwidth]{diagram4}} \sim \text{\includegraphics[width=0.4\textwidth]{diagram5}} \]
To make a group $B_{\infty}$ out of ordinary diagrams, use completion:

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \quad \sim \quad \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{array}
, \quad i.e., \quad \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \quad \sim \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\]

To make a group out of parenthesized diagrams, use similar completions:

\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\end{array} \quad \sim \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}
\]
To make a group $B_\infty$ out of ordinary diagrams, use completion:

\[
\begin{array}{ccc}
\bullet \bullet \bullet & \sim \sim \sim & \bullet \bullet \bullet
\end{array}
\]
\[\text{, i.e.,} \]

\[
\begin{array}{ccc}
\bullet \bullet \bullet \bullet \bullet & \sim \sim \sim \sim \sim \sim
\end{array}
\]

To make a group out of parenthesized diagrams, use similar completions:

\[
\begin{array}{ccc}
\bullet \bullet \bullet & \sim \sim \sim & \bullet \bullet \bullet \bullet \bullet
\end{array}
\]

\[\sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sin
To make a group $B_\infty$ out of ordinary diagrams, use completion:

$$\rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow , \text{ i.e., } \bullet \bullet \bullet \bullet \rightsquigarrow \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$$

To make a group out of parenthesized diagrams, use similar completions:

$$\rightsquigarrow \rightsquigarrow$$

Index positions by sequences of integers (or infinitesimals):

$$\rightsquigarrow$$

1 2 3

$$\bullet \bullet \bullet$$
• To make a group $B_\infty$ out of ordinary diagrams, use completion:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{ordinary_diagram} \\
\sim \sim \sim \sim \sim
\end{array}
\quad , \quad i.e.,
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{completed_diagram} \\
\sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim
\end{array}
\]

• To make a group out of parenthesized diagrams, use similar completions:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{parenthesized_diagram} \\
\sim \sim
\end{array}
\quad , \quad Index \ positions \ by \ sequences \ of \ integers \ (or \ infinitesimals):

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{indexed_diagram} \\
\sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim
\end{array}
\]

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{indexed_diagram2} \\
\sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim
\end{array}
\]

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{indexed_diagram3} \\
\sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim
\end{array}
\]
To make a group $B_\infty$ out of ordinary diagrams, use completion:

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

To make a group out of parenthesized diagrams, use similar completions:

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

Index positions by sequences of integers (or infinitesimals):

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\nearrow & & & \\
\cdots & & & \\
\searrow & & & \\
\end{array}
\]
To make a group $B_\infty$ out of ordinary diagrams, use completion:

\[
\begin{array}{c}
\includegraphics{example1} \\
\sim\Rightarrow \includegraphics{example2}
\end{array}
\]

, i.e.,

\[
\begin{array}{c}
\includegraphics{example3} \\
\sim\Rightarrow \includegraphics{example4}
\end{array}
\]

To make a group out of parenthesized diagrams, use similar completions:

\[
\begin{array}{c}
\includegraphics{example5} \\
\sim\Rightarrow \includegraphics{example6}
\end{array}
\]

\[
\begin{array}{c}
\sim\Rightarrow \includegraphics{example7} \\
\sim\Rightarrow \includegraphics{example8}
\end{array}
\]

\[
\begin{array}{c}
\sim\Rightarrow \includegraphics{example9} \\
\sim\Rightarrow \includegraphics{example10}
\end{array}
\]

\[
\sim\Rightarrow \includegraphics{example11}
\]

Index positions by sequences of integers (or infinitesimals):

\[
\begin{array}{c}
\sim\Rightarrow \includegraphics{example12} \\
\sim\Rightarrow \includegraphics{example13}
\end{array}
\]

\[
\begin{array}{c}
\sim\Rightarrow \includegraphics{example14} \\
\sim\Rightarrow \includegraphics{example15}
\end{array}
\]

\[
\sim\Rightarrow \includegraphics{example16}
\]

\[
\sim\Rightarrow \includegraphics{example17}
\]
• To make a group $B_\infty$ out of ordinary diagrams, use completion:

```
\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} \quad \sim \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]
```

\[\sim, \text{ i.e.,} \quad \bullet \bullet \bullet \bullet \sim \bullet \bullet \bullet \bullet \]

• To make a group out of parenthesized diagrams, use similar completions:

```
\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\end{array} \sim \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]
```

\[\sim \text{ Index positions by sequences of integers (or infinitesimals):} \]

```
\[
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \sim \begin{array}{ccc}
1 & 1,2 & 2,2,2,3 \sim \begin{array}{ccc}
1 & 2 & 3 \\
1+\epsilon & 2+\epsilon & \sim \begin{array}{ccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{array}
\]
```

• Definition: $B := \{ \text{parenthesized braid diagrams} \}/ \text{isotopy}$. 

• The following relations hold in $B$:
The following relations hold in $B$:

- **Commutation:**
  
  \[ \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \sigma_i a_j = a_j \sigma_i, \quad \text{for } j \geq i + 2, \]
The following relations hold in $B$:  

**commutation:**  
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**"Thompson":**  
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**Notation:** $\tilde{B} :=$ the group presented by $\uparrow$;
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- **Notation:** $\widetilde{B} :=$ the group presented by $\uparrow$; know $\widetilde{B} \rightarrow B$; want: $\widetilde{B} \simeq B$. 
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generators: $\sigma_1, \sigma_2, a_1, a_2$

$e.g., \sigma_3 = a_1^{-1} \sigma_2 a_1 = \cdots$
ALGEBRAIC PROPERTIES OF $\tilde{B}$

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  - Generators: $\sigma_1, \sigma_2, a_1, a_2$
  - Example: $\sigma_3 = a_1^{-1} \sigma_2 a_1$
  - The monoid with presentation...

- Proposition: The group $\tilde{B}$ is a group of fractions for the monoid $\tilde{B}^+$, and
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  of the type “dilatation + braid + contraction”.

• Also: locally Garside structure for $\widetilde{B}^+$.
  
  w.r.t. balanced generators $A_\alpha, \Sigma_\alpha$ indexed by binary addresses
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• Proposition: Let \( \text{sh} : \sigma_i \mapsto \sigma_{i+1}, a_i \mapsto a_{i+1} \) for each \( i \). For \( g, h \) in \( \tilde{B} \), let

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g \ast h := g \cdot \text{sh}(h) \cdot \sigma_1 \cdot \text{sh}(h)^{-1},
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g \circ h := g \cdot \text{sh}(h) \cdot a_1.
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Then $(\tilde{B}, \ast, \circ)$ is an augmented LD-system.
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\[
\begin{align*}
\begin{cases}
  x \ast (y \ast z) &= (x \ast y) \ast (x \ast z) : \text{self-distributivity, "LD"} \\
  x \ast (y \ast z) &= (x \circ y) \ast z \\
  x \ast (y \circ z) &= (x \ast y) \circ (x \ast z)
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\[ x \times y \]

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Classical examples:
- $x \ast y = y$ leads to $B_n \hookrightarrow S_n$;
- $x \ast y = x y x^{-1}$ leads to $B_n \leftrightarrow \text{Aut}(F_n)$;
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Here: $(\widetilde{B}, \ast)$ eligible for colouring.
• Proposition: $\mathcal{B} \simeq \tilde{\mathcal{B}}$, i.e., ... is a presentation for $\mathcal{B}$. 
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\[\uparrow\]

under \( \text{eval} \), the action of \( D(w) \) on \( \tilde{B} \) becomes a multiplication. \( \Box \)
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Each element of \(B\) generates a free subsystem.
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MORE ABOUT THE LD-STRUCTURE OF $B$

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\(\leftrightarrow\) \(B\) is connected with the “geometry group” of the algebraic laws ALD.
Well-known: $B_n \simeq \text{MCG}(D_n) \leftrightarrow \text{Aut}(F_n)$.
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Here: $D_n \mapsto$ sphere with a Cantor set of punctures
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Here: $D_n \mapsto$ sphere with a Cantor set of punctures $S^2 \setminus \text{Cantor}$
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$S^2 \setminus \text{Cantor}$ ← a continuous gap with countably many bridges indexed by dyadic numbers
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$S^2 \setminus \text{Cantor}$

$\sigma_i$ Dehn half-twist: exchanging $U_i$ and $U_{i+1}$

$\alpha_i$ dilatation: expanding $U_i$ into $U_i \sqcup U_{i+1}$
Lemma: $B \to \text{MCG}(S^2 \setminus \text{Cantor})$. 
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Look at action on $\pi_1$. 
Lemma: \( B \rightarrow \text{MCG}(S^2 \setminus \text{Cantor}) \).

\[ \Rightarrow \text{Look at action on } \pi_1. \]

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ACTION ON THE FUNDAMENTAL GROUP

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basis indexed by finite sequences of integers
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ARTIN REPRESENTATION

\[ x_1 x_1 x_1 \]
\[ x_2 x_2 x_2 \]
\[ x_3 x_3 x_3 x_1,1 x_1,1 x_1,1 x_1,1 \]
\[ x_1,2 x_1,2 x_1,2 x_1,2 \]
\[ x_2,1 x_2,1 x_2,1 x_2,1 \]
\[ \sigma_1 \]
\begin{equation}
\{ x_{1,s} \mapsto x_1 x_{2,s} x_{1}^{-1}, \ x_{2,s} \mapsto x_{1,s}, \ x_{j,s} \mapsto x_{j,s} \ (j \geq 3) \}
\end{equation}
ARTIN REPRESENTATION

\[
x_1 \xleftrightarrow{a_1} x_1 x_2 \xleftrightarrow{\sigma_1} x_1 x_2 x_1^{-1}
\]

\[
\begin{cases}
x_{1,s} \mapsto x_{1} x_{2,s} x_{1}^{-1} \\
x_{2,s} \mapsto x_{1,s} \\
x_{j,s} \mapsto x_{j,s} \quad (j \geq 3)
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\[ x_1, s \mapsto x_1 x_2, s x_1^{-1} \]
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Lemma: $B \rightarrow$ subgroup of $\text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_\infty)$. 
Lemma: \( B \rightarrow \text{subgroup of } \text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_{\infty}). \)

Theorem: The above mappings are embeddings.
• Lemma: $B \rightarrow$ subgroup of $\text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_\infty)$.

• Theorem: The above mappings are embeddings.

~\Rightarrow~ The Artin representation of $B$ is faithful.
• **Lemma:** \( B \rightarrow \text{subgroup of } \text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_\infty). \)

• **Theorem:** The above mappings are **embeddings**.

  \[ \leadsto \text{The Artin representation of } B \text{ is faithful.} \]

• **Sketch of proof** ( \[ \leadsto \text{uses the LD-structure again} \):
• **Lemma:** $B \rightarrow$ subgroup of $\text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_{\infty})$.

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• **Sketch of proof** ( $\leadsto$ uses the LD-structure again):
  Want to show: If $w$ is a word with at least one $\sigma_1$ and no $\sigma_1^{-1}$, then the automorphism $\tilde{w}$ associated with $w$ moves some $x_s$ $\leadsto$ hence $\neq \text{id}$. 
• **Lemma:** \( B \rightarrow \) subgroup of \( \text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_\infty) \).

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Want to show: If \( w \) is a word with at least one \( \sigma_1 \) and no \( \sigma_1^{-1} \), then the automorphism \( \tilde{w} \) associated with \( w \) moves some \( x_s \) \( \implies \) hence \( \neq \text{id} \).

Key point: \( \tilde{w} \) can be read from \( w \) by colouring trees;

(similar to the Hurwitz action of a braid)
• Lemma: $B \rightarrow \text{subgroup of } \text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_\infty)$. 

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  Key point: $\tilde{w}$ can be read from $w$ by colouring trees; 

  (similar to the Hurwitz action of a braid)

  Then use Larue's method: if $w$ contains at least one $\sigma_1$ and no $\sigma_1^{-1}$, 

  then $\tilde{w}(x_1)$ finishes with $x_1^{-1}$. 

  (control reductions)
• Lemma: $B \rightarrow \text{subgroup of } \text{MCG}(S^2 \setminus \text{Cantor}) \rightarrow \text{Aut}(F_{\infty})$.

• Theorem: The above mappings are embeddings.  

  $\implies$ The Artin representation of $B$ is faithful.

• Sketch of proof ( $\implies$ uses the LD-structure again): 

  Want to show: If $w$ is a word with at least one $\sigma_1$ and no $\sigma_1^{-1}$, then 
  
  the automorphism $\tilde{w}$ associated with $w$ moves some $x_s$  $\implies$ hence $\neq \text{id}$.

  Key point: $\tilde{w}$ can be read from $w$ by colouring trees; 

  (similar to the Hurwitz action of a braid)

  Then use Larue's method: if $w$ contains at least one $\sigma_1$ and no $\sigma_1^{-1}$, 

  then $\tilde{w}(x_1)$ finishes with $x_1^{-1}$. 

  (control reductions)  

$\blacksquare$
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