• General principle (Brieskorn, Alexander): Colour the arcs of a braid or a link diagram to extract information about the braid or the link.

  $x \ast (y \ast z) = (x \ast y) \ast (x \ast z)$.

  → algebraic translation of Reidemeister move of type III.

  → Use various types of self-distributive operations (classical and non-classical) to various applications.

• Aim: To show how various colouring techniques can be used.
Consider a standard braid or link diagram $D$:

Attach colours from a set $S$ to the arcs of $D$, and propagate them along the arcs.

Not much to learn if colours never change;

More interesting if colours may change:

Fix rules for crossings:
Invariance under isotopy

- Want information about the braid or the link represented by the diagram, not about the diagram require invariance under isotopy.
- Case of braids:
  - Standard generators:
    \[ \sigma_i : \quad 1 \quad 2 \quad \cdots \quad i \quad i+1 \quad \cdots \quad n \]
    \[ \sigma_i : \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \times \quad \downarrow \quad \cdots \quad \downarrow \]
  - Standard presentation for the braid group \( B_n \), and the braid monoid \( B_n^+ \):
    \[ \left\langle \sigma_1, \ldots, \sigma_{n-1} ; \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i - j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1 \end{cases} \right\rangle \]
  - Then: invariance under isotopy = compatibility with braid relations.
Case of positive braids

• Fact.- Colouring is compatible with isotopy iff $\ast$ satisfies Identity LD:

$$x \ast (y \ast z) = (x \ast y) \ast (x \ast z).$$

(1)

Proof:

• Def.— $(S, \ast)$ is an \textbf{LD-system} if $\ast$ satisfies (1).
Case of arbitrary braids

- Fact.- Colouring is compatible with isotopy iff $\ast$ satisfies Identity LD, plus

$$x \ast (x \ast y) = x \ast (x \ast y) = y.$$  \hspace{1cm} (2)

Proof:

$\Rightarrow \bar{\ast}$ is a left inverse for $\ast$: left translations rel to $\ast$ and $\bar{\ast}$ are bijections,

$\Rightarrow$ left cancellation is allowed for $\ast$ and $\bar{\ast}$

$\Rightarrow \ast$ determines $\bar{\ast}$: $x \ast y = $ the unique $z$ satisfying $x \ast z = y$.

- Def.– $(R, \ast, \bar{\ast})$ is a rack if $\ast$ satisfies (1) plus (2).
Case of links

- Invariance under isotopy = compatibility with Reidemeister moves
- Fact. - Colouring is compatible with Reidemeister moves iff $\ast, \, \bar{\ast}$ satisfies the rack identities, plus

$$x \ast x = x.$$  (3)

Proof:

\[ y \ast (y \bar{\ast} x) = x \ast x \]

\[ y \bar{\ast} x = x \ast (x \bar{\ast} y) \]
• Def. – \((Q, *, \bar{*})\) is a quandle if \(*\) satisfies (1), (2), (3).
Two ways of using colourings

Braids are open, knots and links are closed \(\Rightarrow\) different ways of using colourings.

- Braids: The Hurwitz action of braids on sequences of colours.

  - Fix one rack \((R, \ast)\), and use it to colour every braid \(b\): \(\Rightarrow\) \(b\) defines a map of \(R^n\) to itself.

\[
\begin{array}{ccccccc}
  x_1 & x_2 & x_3 & \ldots & \in R^n \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  b & & & \ldots & \mapsto \rho_b : R^n \to R^n \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  y_1 & y_2 & y_3 & \ldots & \in R^n
\end{array}
\]

- Def. – For \((R, \ast, \ast)\) a rack, put \(x \ast \varepsilon = x\) (for \(\varepsilon\) = empty word), and

\[
\begin{align*}
  x \ast (\sigma_i w) &= (x_1, \ldots, x_{i-1}, x_i \ast x_{i+1}, x_i, x_{i+2} \ldots) \\
  x \ast (\sigma^{-1}_i w) &= (x_1, \ldots, x_{i-1}, x_{i+1}, x_i \ast x_{i+1}, x_{i+2} \ldots)
\end{align*}
\]

- Proposition.- (Brieskorn) For each LD-system \((S, \ast)\) one obtains an action of \(B_n^+\) on \(S^n\).
  For each rack \((R, \ast, \ast)\) one obtains an action of \(B_n\) on \(R^n\).
Links: pushing the colours leads to obstructions

- quotient of the initial quandle (depending on the link)
- invariant of that link

\[ L = \hat{b} \]

\[ Q_L = \frac{Q}{(y_1 = x_1, y_2 = x_2, \ldots)} \]

- the more general the quandle, the most powerful the invariant.
- fundamental quandle: \( Q_L \) for \( Q \) free on \( n \) generators if \( L \) closure of an \( n \) strand braid.

Proposition.- (Joyce, Matveev) The fundamental quandle is a complete invariant of the isotopy type up to a mirror symmetry.

(BUT problem: how to compute \( Q_L \)?)
Example 1: trivial rack

- Take $S = \text{any set}$, and

$$x \ast y = y, \quad x \bar{\ast} y = y.$$ 

$\Rightarrow$ a rack, even a quandle; 
$\Rightarrow$ amounts to not changing colours.

- For braids: leads to

$$x \bullet b = \text{perm}(b)(x)$$

where $\text{perm}(b)$ is the permutation associated with $b$. 
$\Rightarrow$ Here, the Hurwitz action leads to 

$$\text{perm} : B_n \longrightarrow \mathfrak{S}_n.$$ 

- For links: identifying output colours with input colours yields a quotient with $k$ elements for a link $L$ with $k$ components.
Example 2: shift rack

• Take \( \mathbb{Z} = \) the integers, and
\[
x \ast y = y + 1, \quad x \bar{\ast} y = y - 1.
\]
\( \Rightarrow \) a rack, not a quandle \((0 \ast 0 = 1)\).

• For braids: leads to
\[
\sum (x \cdot b) = \sum x + \text{sum}(b)
\]
where \( \text{sum}(b) \) is the exponent sum of \( b \).
\( \Rightarrow \) Here, the Hurwitz action leads to the augmentation homomorphism
\[
\text{sum} : B_n \rightarrow (\mathbb{Z}, +)
\]
mapping every \( \sigma_i \) to \( 1 \).
Example 3: Alexander rack

- Take for $E$ a $\mathbb{Z}[t, t^{-1}]$-module, and
  \[
  x * y = (1 - t)x + ty, \quad x \star y = (1 - t^{-1})x + t^{-1}y
  \]
  a rack, even a quandle.

- For braids: leads to
  \[
  \mathbf{x} \cdot b = \mathbf{x} \times r_B(b)
  \]
  where $r_B(b)$ is an $n \times n$ matrix associated with $b$.

  Here, the Hurwitz action gives a linear representation
  \[
  r_B : B_n \to GL_n(\mathbb{Z}[t, t^{-1}])
  \]
  the (unreduced) Burau representation.

- For links: quotienting under $\mathbf{x} \cdot b = \mathbf{x}$ gives the Alexander ideal
  hence the Alexander polynomial.
Example 4: conjugacy rack

- Take for $F_n$ a the free group based on \( \{x_1, \ldots, x_n\} \), and
  \[
  x \ast y = x y x^{-1}, \quad x \bar{\ast} y = x^{-1} y x
  \]
  a rack, even a quandle.

- For braids: Define \( y_1, \ldots, y_n \) by
  \[
  (x_1, \ldots, x_n) \bullet b = (y_1, \ldots, y_n).
  \]
  Then \( \varphi(b) : x_i \mapsto y_i \) is an automorphism of \( F_n \).
  Here the Hurwitz action gives Artin’s representation
  \[
  \varphi : B_n \to Aut(F_n).
  \]

- For links: quotienting under \( x \bullet b = \hat{x} \) defines a group associated with the closure of \( b \)
  the fundamental group of the complement of \( \hat{b} \), via its Wirtinger presentation.
Example 5: free racks

Are there many more different types of racks?

\(\Rightarrow\) NO: conjugacy racks are close to free racks, i.e., the most general possible racks.

Let \(G\) be a group and \(X \subseteq G\); on \(G \times X\) take

\[(a, x) \ast (b, y) = (axa^{-1}b, y), \quad (a, x) \bar{\ast} (b, y) = (ax^{-1}a^{-1}b, y).\]

\(\bullet\) Fact.- This is a rack, and, for \(G\) free based on \(X\), the rack is free.

\(\Rightarrow\) close to conjugacy (‘first half of conjugacy words’),

\(\Rightarrow\) in particular, always nearly idempotent:

\[x \ast y = (x \ast x) \ast y.\]

\(\bullet\) Questions.— 1. Does there exist LD-systems of a different type?

\(\quad\) (in particular where left division has no cycle)

2. (If so) Can one use them to colour braid or link diagrams?
Example 6: injection bracket

- Take $I_\infty$ = the set of all injective, non-bijective mappings of $\mathbb{N}$ into itself, and

$$f \ast g(n) = \begin{cases} fgf^{-1}(n) & \text{for } n \text{ in the image of } f, \\ n & \text{otherwise}. \end{cases}$$

\[ \Longrightarrow \] An LD-system in which $x \ast y = (x \ast x) \ast y$ is false

(and whose presentation is unknown).
At the end of the 1980’s: new, completely different LD-systems coming from Set Theory
   not directly useful here, but gave (strong) motivation for further study.

   Arbitrary LD-systems are OK for positive braid diagrams, but
   Problem for arbitrary diagrams
      (Can be coloured, but no uniqueness or invariance).

   Technical detour: braid word reversing
Let $\sigma = \sigma_1, \sigma_2, \ldots$. Define $f : \sigma \times \sigma \to \sigma^*$ (the words on $\sigma$) by

$$f(\sigma_i, \sigma_j) = \begin{cases} 
\sigma_j & \text{for } |i - j| \geq 2, \\
\sigma_j \sigma_i & \text{for } |i - j| = 1, \\
\varepsilon & \text{for } i = j.
\end{cases}$$

The presentation of $B_n$ consists of all relations

$$\sigma_i f(\sigma_i, \sigma_j) = \sigma_j f(\sigma_j, \sigma_i). \quad (\ast)$$

Now $\ast$ also implies

$$\sigma_i^{-1} \sigma_j = f(\sigma_i, \sigma_j) f(\sigma_j, \sigma_i)^{-1}.$$

When we replace a subword of the form $\sigma_i^{-1} \sigma_j$ with the corresponding $f(\sigma_i, \sigma_j) f(\sigma_j, \sigma_i)^{-1}$ in a braid word, we obtain an equivalent word.

- Def.– Say that a braid word $w$ is right reversible to $w'$ if one can transform $w$ into $w'$ in this way (i.e., by iteratively pushing the negative letters to the right and the positive to the left).

If $w$ is right reversible to $w$, then $w$ and $w'$ are equivalent, but no converse (of course).
A partial Hurwitz action

... nevertheless, partial converse implication:

- Proposition.- If $u, v$ are positive braid words, then $u$ and $v$ are equivalent (i.e., represent the same braid) if and only if $u^{-1}v$ is right reversible to the empty word.

Let $(S, \ast)$ be a left cancellative LD-system; for each sequence of input colours $x$ and each braid word $w$,
- there exists at most one colouring of (the diagram coded by) $w$ starting with $x$,
- if so, there exists exactly one colouring with the same input and output colours for each word $w'$ such that $w$ is right reversible to $w'$.

A partial action of $B_n$ on $S^n$: for $x$ a sequence of colours and $b$ a braid,
- $x \bullet b$ need not exist, but
- there always exists at least one sequence $x$ s.t. $x \bullet b$ exists, and
- $x \bullet b$ is uniquely determined when it exists.
Free LD-systems

- Def. – **D** = the free LD-system on one generator.

  \[ \Rightarrow D \text{ consists of all expressions } g, g \ast g, g \ast (g \ast g), \ldots \text{ with LD-equivalent expressions identified}; \]

  \[ \Rightarrow \text{similar to } Z_+ \text{ when self-distributivity } x(yz) = (xy)(xz) \text{ replaces associativity } x(yz) = (xy)z. \]

  \((Z_+ \text{ is the free semigroup on one generator})\)

- In the case of **Z_+**: \((\exists z)(y = x + z)\) defines a **linear ordering**;

  \[ \Rightarrow \text{similar in the case of } D \text{ (but more difficult to prove...)}:\]

  \[ \text{• Proposition.- The transitive closure } \sqsubseteq \text{ of the relation } (\exists z)(y = x \ast z) \text{ is a linear ordering on } D. \]

  \[ \Rightarrow D \text{ is left cancellative} \]

  \[ \Rightarrow \text{Use } D \text{ to colour braids, and its ordering to order them:} \]

  \[ \text{• Proposition.- For } b_1, b_2 \text{ in } B_n, \text{ say that } b_1 < b_2 \text{ is true if } x \bullet b_1 \sqsubseteq^{\text{Lex}} x \bullet b_2 \text{ holds for some } x \text{ in } D^n. \text{ Then } < \text{ is a linear ordering on } B_n \text{ compatible with multiplication on the left.} \]
An intrinsic construction of the braid ordering

⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒

Intrinsic construction of the previous braid ordering? (not appealing to \(D\))

• \(\partial = \text{shift endomorphism of } B_\infty\), i.e., \(\partial : \sigma_i \mapsto \sigma_{i+1}\) for each \(i\).

• Def.— A braid \(b\) is \(\sigma_1\)-positive if, among all possible expressions of \(b\), there is at least one in which \(\sigma_1\) occurs, but \(\sigma_1^{-1}\) does not. A braid \(b\) is \(\sigma\)-positive if it is \(\partial^k b_0\) for some \(\sigma_1\)-positive braid \(b_0\).

⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒

Example: \(\sigma_1 \sigma_2 \sigma_1^{-1}\) is \(\sigma_1\)-positive: \(\sigma_1 \sigma_2 \sigma_1^{-1} = \sigma_2^{-1} \sigma_1 \sigma_2\): one \(\sigma_1\), no \(\sigma_1^{-1}\).

⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒

• Proposition.— The relation “\(b_1^{-1} b_2\) is \(\sigma\)-positive” is a linear ordering on \(B_n\), and it coincides with the ordering coming from \(D\).

⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒⇒

Two points to prove (and we shall do it using colourings):

• **Property A**: A \(\sigma_1\)-positive braid is not trivial;
• **Property C**: Every braid is \(\sigma_1\)-positive, or \(\sigma_1\)-negative, or \(\sigma_1\)-free.

\(b\) is \(\sigma_1\)-negative = \(b^{-1}\) is \(\sigma_1\)-positive; \(b\) is \(\sigma_1\)-free = \(b\) belongs to the image of \(\partial)\)
Consider a $\sigma_1$-positive diagram (want to prove it does not represents $1$)

\[ \Rightarrow \text{put colours from } D: \]

\[ \begin{array}{c}
\bullet 0 \\
\bullet 1 \\
\bullet 2 \\
\bullet 3 \\
\end{array} \]

By construction: $\bullet_0 \sqsubseteq \bullet_1 \sqsubseteq \bullet_2 \sqsubseteq \ldots$, hence $\bullet_p \neq \bullet_0$.

(Recall: $z \sqsubseteq z'$ is the transitive closure of $(\exists y)(z' = z * y)$)
A self-distributive operation on braids

• Def.– For $b_1, b_2$ in $B_\infty$, define $b_1 \ast b_2 = b_1 \cdot \partial b_2 \cdot \sigma_1 \cdot \partial b_1^{-1}$.

$\Rightarrow$ Example: $1 \ast 1 = \sigma_1$, $1 \ast (1 \ast 1) = \sigma_2 \sigma_1$, $(1 \ast 1) \ast 1 = \sigma_1^2 \sigma_2^{-1}$, ...

• Fact.– $(B_\infty, \ast)$ is a left cancellative LD-system.

$\Rightarrow$ One can use $B_\infty, \ast$ to colour braids.
Def. – A braid $b$ is called **special** if it belongs to the closure of $\{1\}$ under $\ast$.

Fact. – For $b$ special, $(1, 1, \ldots) \ast b = (b, 1, 1, \ldots)$ (“special braids are self-colouring”).

**Inductive proof:**
Proof of Property $C$

(Property $C$: every braid is $\sigma_1$-positive, $\sigma_1$-negative or $\sigma_1$-free)

$\bullet$ Fact.- Every braid $b$ in $B_n$ admits a decomposition

\[ b = \partial^{n-1} b_n^{-1} \cdot \ldots \cdot \partial b_2^{-1} \cdot b_1^{-1} \cdot b_1' \cdot \partial b_2' \cdot \ldots \cdot \partial^{n-1} b_n'. \]

where $b_1, \ldots, b_n, b_1', \ldots, b_n'$ are special.

$\bullet$ Fact.- If $b, b'$ are special braids, then $b^{-1} b'$ is either $\sigma_1$-positive, or $\sigma_1$-negative, or equal to 1.

(easy from properties of $D$ and $\ast$: $b' = b \ast x$ implies $b^{-1} b' = \partial(x) \cdot \sigma_1 \cdot \partial(b^{-1})$.)

$\Rightarrow$ Property $C$ (other proofs known, but none much easier).
A few open questions

• Property $S$
A non-trivial property of the braid ordering: For every braid $b$, one has $b\sigma_i > b$ for each $i$.

Question.– Is there a natural proof of Property $S$ based on diagram colourings?

• Handle reduction
An efficient solution to the isotopy problem of braids: A $\sigma_i$-handle is a braid word of the form $\sigma_i^e w \sigma_i^{-e}$ with $e = \pm 1$ and $w$ containing no $\sigma_j^{\pm 1}$ with $j \leq i$ and, in addition, not containing both $\sigma_{i+1}$ and $\sigma_{i+1}^{-1}$.

Reducing a handle means deleting the initial and final $\sigma_i^e$ and substituting each $\sigma_{i+1}^d$ with $\sigma_{i+1}^{-e} \sigma_i^d \sigma_{i+1}^e$.

The braid ordering forces convergence (and practical efficiency), but

Question.– What is the complexity of handle reduction?

• Special braids (those that can be obtained from 1 using $x \ast y = x \cdot \partial(y) \cdot \sigma_1 \cdot \partial(x^{-1})$)

Question.– How many special braids lie in $B_n$?
• **Twisted conjugacy**

The self-distributive operation $\ast$ on $B_\infty$ is a twisted version of conjugacy.

⇝ Question.– Can one replace the standard conjugacy operation with its twisted version involving $\ast$ in the design of braid-based cryptosystems?

⇝ Question.– Is there an algorithm deciding whether two braids $b, b'$ are twisted-conjugate?

⇝ Question.– Is there a constructive way to recover $b$ from $b \ast 1$?

• **Arbitrary LD-systems**

⇝ Question.– Can one use arbitrary left cancellative LD-systems, in particular those that are not racks, to colour links diagrams?

⇝ Question.– Can one use arbitrary LD-systems, in particular those that are not left cancellative, to colour braid (or link) diagrams?
A new (seemingly very interesting) group that extends both Artin’s group $B_\infty$ and Richard Thompson’s group $F$.

$$F = \langle a_1, a_2, \ldots; a_i a_j = a_{j+1} a_i \text{ for } j > i \rangle.$$ 

Braid diagrams replaced with tree diagrams; connected with associativity, and with piecewise linear diffeomorphisms of $(0, 1)$.

- **Def.** $B_T = \left\langle \{\sigma_1, \sigma_2, \ldots\}; \{\text{Artin’s relations } + \sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i\} \right. \left. \text{Thompson’s relations } + \sigma_{i+1} \sigma_i a_{i+1} = a_i \sigma_i\right\rangle$

  includes $B_\infty$ and $F$;

- **Def.** For $b_1, b_2$ in $B_T$, define $b_1 \ast b_2 = b_1 \cdot \partial b_2 \cdot \sigma_1 \cdot \partial b_1^{-1}$.

- **Fact.** $(B_T, \ast)$ is a left cancellative LD-system.

- **Question.** What can one do with $B_T$-colourings?
  (prove that $B_T$ is orderable)
Thompson's braid group as a mapping class group

\[ x_1 \]

\[ x_{1,1} \]

\[ x_{1,1,1} \]

\[ x_2 \]

\[ x_{2,1} \]

\[ x_3 \]

\[ S^2 \setminus \text{Cantor set} \]

\[ \leadsto \]

\[ \rightsquigarrow \text{a Cantor river with countably many bridges} \]
Artin representation of $B_T$

\( \sigma_1 : \)

\( a_1 : \)

\( x_1 \to x_1 x_2 x_1^{-1}, \quad x_2 \to x_1, \quad x_3 \to x_3.\) \( x_{1,1} \to x_{1,2} x_{1,1} x_{1}^{-1}, \quad x_{2,1} \to x_{1,1}.\)

\( x_1 \to x_1 x_2, \quad x_2 \to x_3, \quad x_3 \to x_4.\) \( x_{1,1} \to x_1, \quad x_{2,1} \to x_{3,1}.\)

\( \leadsto \) faithful representation of $B_T$:
The Laver tables

A distinguished family of \textit{finite LD-systems}.

Construct a left self-distributive operation on $\{1, 2, \ldots, N\}$ from

At most one solution,

can be completed for $N = 2^n$ only

$A_n$, the $n$th Laver table, a finite LD-system with $2^n$ elements

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Question.– Can one use the Laver tables to colour diagrams? (enough complicated to be promising)
References
