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BRAID ORDER, SETS, AND KNOTS

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Abstract. We survey two of the many aspects of the standard braid order, namely its set theoretical roots, and the known connections with knot theory, including results by Netsvetaev, Malyutin, and Ito, and very recent work in progress by Fromentin and Gebhardt.

It has been known since 1992 [7, 8] that Artin's braid groups B_n are left-orderable, by an ordering that has several remarkable properties. In particular, it was proved in [23] that its restriction to the braid monoid B_n^+ is a well-ordering. Many subsequent results were established using different approaches, and the subject has developed so as to become the whole content of the monograph [11]. The first main point is that many different approaches, some of them algebraic or combinatorial, others geometrical of topological, developed by a number of researchers, in particular Burckel, Dynnikov, Fenn, Fromentin, Funk, Greene, Larue, Rolfsen, Rourke, Short, Wiest, and the author, lead to one and the same distinguished ordering on braid groups, the so-called Dehornoy ordering, hereafter referred as the *D-ordering*. The second main point is that the family of all left-invariant braid orderings turns out to be an interesting space in which the D-ordering plays a significant role, as shown in works by Clay, Ito, Navas, Rolfsen, Short, Wiest.

The purpose of the current survey paper is not to repeat the material that is developed in [11], nor even to give a comprehensive introduction to that text, but rather to point out some aspects that are not, or not fully, described there, namely the set-theoretical roots of the D-ordering, as well as its known connections with knot theory. These aspects are alluded to in Sections III.2 and IV.5 of [11], but the latter is a quite short introduction, whereas the former is already obsolete due to several recent developments.

Acknowlegment

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1. Connections with Set Theory

It turns out that the first proof of the result that Artin's braid groups are orderable groups and the construction of the specific D-ordering of braids stem from questions of Set Theory involving so-called large cardinal axioms. Here we shall briefly explain this connection. However, before starting, let us insist that the connection is historical rather than logical, as no set-theoretical axiom has ever been used in the construction of the braid order, the latter appearing precisely at the moment when set-theoretical axioms disappeared from the landscape.

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1.1. **Braid groups.** We recall that, for $n \geqslant 2$, Artin's braid group B_n is the group that admits the finite presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} \quad |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} \quad |i-j| = 1 \end{array} \right\rangle. \tag{1}$$

The elements of B_n are called *n-strand braids*. The braid group on infinitely many strands, denoted B_∞, is defined by a presentation with infinitely many generators $\sigma_1, \sigma_2, \ldots$ subject to the same relations. The identity mapping on $\{\sigma_1, \ldots, \sigma_{n-1}\}$ extends into an injective homomorphism of B_n to B_{n+1} and, therefore, we can identify B_n with the subgroup of B_∞ generated by $\sigma_1, \ldots, \sigma_{n-1}$.

According to the above definition, every braid admits decompositions in terms of the generators σ_i and their inverses. A word on the letters $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses is called an *n-strand braid word*. If the braid β is the equivalence class of the braid word w, we say that *w represents* β, or is an *expression* of β. We say that two braid words are *equivalent* if they represent the same braid, *i.e.*, if they are equivalent with respect to the least equivalence relation that contains the relations of (1) and is compatible with multiplication.

It is well-known that B_n can be interpreted as the group of isotopy classes of n-strand braid diagrams [1]. Under this correspondence, the generator σ_i corresponds to the (isotopy class of the) elementary diagram in which the strand initially at position $i+1$ crosses over the strand initially at position i, see Figure 1.

FIGURE 1. Interpretation of σ_i as an elementary braid diagram—here the strands have an overall vertical direction

It is also well-known [1] that B_n can be interpreted as the mapping class group of a disk with n punctures. Under this interpretation, σ_i corresponds to the (isotopy class of the) half Dehn twist that exchanges the ith and the $i+1$st punctures clockwise, and keeps the other punctures as well as every point on the boundary circle fixed, see Figure 2.

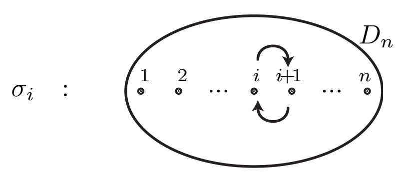


FIGURE 2. Interpretation of σ_i as a Dehn half-twist in an n punctured disk

1.2. **The D-ordering of braids.** We start from the following notion.

Definition. We say that a nonempty braid word w is *σ-positive* if all letters σ_i with minimal i that occur in w have positive exponents.

For instance, the braid word $\sigma_1\sigma_2\sigma_1^{-1}$ is not σ-positive, since the letter σ_i with minimal i occurring in that word, namely σ_1, occurs both with exponent $+1$ and with exponent -1. On the other hand, the braid word $\sigma_2^{-1}\sigma_1\sigma_2$—which turns out to represent the same braid as the previous word—is σ-positive since, in this word, σ_1 occurs with exponent $+1$ only.

Saying that a braid word w is σ-positive means that, in the braid diagram encoded by w, all bottom crossings are positive—according to the convention that the strands are drawn horizontally and numbered from bottom up, see Figure 3.

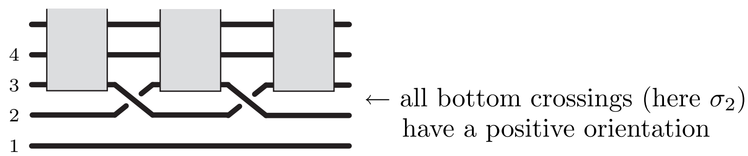


FIGURE 3. A σ-positive braid diagram—here strands are drawn horizontally and numbered from bottom

The main result now is:

Theorem 2. [7, 8] *For β, β' in B_∞, declare that $\beta < \beta'$ holds if $\beta^{-1}\beta'$ can be represented by a σ-positive diagram. Then $<$ is a left-invariant linear ordering on B_∞.*

By definition, an order is called *linear* if any two elements are comparable, and, assuming that the domain is a group, it is called *left-invariant* if it is compatible with multiplication on the left, *i.e.*, $\beta < \beta'$ implies $\alpha\beta < \alpha\beta'$ for every α.

For instance, let us consider the braids $\beta = \sigma_1$ and $\beta' = \sigma_2\sigma_1$. Then we find $\beta^{-1}\beta' = \sigma_1^{-1}\sigma_2\sigma_1$, *i.e.*, the braid word $\sigma_1^{-1}\sigma_2\sigma_1$ is one of the possible expressions of the braid $\beta^{-1}\beta'$. We observed above that this braid word is not σ-positive, nor is its inverse either, so we cannot conclude anything. Now we also observed above that the braid word $\sigma_2\sigma_1\sigma_2^{-1}$ is another expression of the same braid, and that this word is σ-positive. So, by definition, we have $\beta < \beta'$.

It should be clear that, in order to prove Theorem 2, several things are to be proved. We shall not try to do it now, but rather ask

Question 3. *Where does Theorem 2 come from?*

And we claim that the answer is the following set-theoretical result of 1986:

Theorem 4. [5] *If j is an elementary embedding of a self-similar rank, then the LD-structure of* $\mathrm{Iter}(j)$ *implies $\mathbf{\Pi}^1_1$-determinacy.*

Our aim in the rest of this section will be to explain (a little) the meaning of Theorem 4 and, mainly, the connection between the latter (mysterious) statement and the existence of a braid ordering.

1.3. **Braid colorings.** The path from Theorem 4 to Theorem 2 goes through a major auxiliary idea, namely using braid colorings, which directly leads to the self-distributivity law.

Assume that S is a nonempty set and that we wish to use the elements of S as sorts of colours applied to the strands of a braid diagram. The principle we shall use is to attach colors to the left (initial) ends of the strands in a braid diagram, then propagate the colors through the diagram, and compare the right (final) colors with the initial colors to extract information about the braid represented by the diagram.

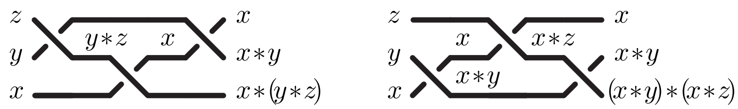
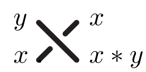


FIGURE 4. Proof of Lemma 5: when the same initial colors are given to the two diagrams, the final colors are the same provided the self-distributivity condition is satisfied

If the colors do not change when strands cross, then the final sequence of colors is simply the image of the initial sequence of colors under some permutation, which corresponds to the standard fact that each n-strand braid induces a well defined permutation of the integers $\{1, ..., n\}$.

A more interesting option is to assume that colors may change at crossings. In terms of complexity, the simplest case corresponds to the case when one of the strands, for instance the back one, keeps its color, whereas the front strand may change, but the new color only depends on the old colors of the two strands involved in the crossing. This case has been been considered by many authors, in particular Joyce [20], Matveev [26], and Brieskorn [3]. It amounts to saying that the set of colors S is equipped with a binary operation $*$, and that colors change according to the rule



Now, as was said above, we wish to get information not about the braid diagram, but about the braid it represents. This means that we want that the final colors only depend on the isotopy class of the considered diagram. According to the presentation of the braid group given in (1), this is true if and only if the colors are not changed when the braid relations are applied. It is obvious to check that the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ with $|i-j| \geqslant 2$ lead to no problem. So the point is to have compatibility with the non-commutative braid relations.

Lemma 5. *Colors are preserved under the braid relations $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ with $|i-j| = 1$ if and only if, for all colors x, y, z, we have*

$$x * (y * z) = (x * y) * (x * z). \tag{6}$$

Equality (6) is called the *left self-distributivity law—LD law* for short—as it asserts that the operation $*$ is distributive with respect to itself on the left. The easy proof of Lemma 5 is shown in Figure 4.

1.4. **LD-systems.** So we are naturally led to look for algebraic systems $(S, *)$ where $*$ is a binary operation on S that satisfies the LD law. Such systems will be called *LD-systems*.

There exist classical examples of LD-systems, and using them to colour the strands of a braid leads to standard results about braid groups. The first famaily consists of the (trivial) operation $x * y = y$ on an arbitrary set S. This amounts to not changing colors in crossings. As was said above, it leads to associate with every n-strand braid a permutation of $\{1, ..., n\}$, thus yielding the well known surjective homomorphism of the braid group B_n onto the symmetric group $\mathfrak{S}_n$.

Another family consists in starting with a group G and using the conjugacy operation $x * y = xyx^{-1}$, which is easily seen to satisfy the LD law. This approach is closely connected with the theory of racks [13] and quandles [20]—yet self-distributive systems are used here in a different spirit, since the idea is not to associate a specific system with each braid or knot, but rather to fix a system and then use it for investigating all braids simultaneously. A particular case is specially interesting, namely when the involved group G is a free group and the initial colors attributed to the strands make a basis of this free group. Then the final colors also make a basis of the free group, thus associating with every n-strand braid β an automorphism $\phi(\beta)$ of the free group of rank n. It can be shown that ϕ is an injective homomorphism, known as the Artin representation of the braid group B_n.

A third family of examples arises when E is an R-module and the operation $x * y = (1-t)x + ty$, where t is a fixed invertible element of the base ring R. Then the final colors are a linear combination of the initial ones, which leads to associating with every n-strand braid an invertible $n \times n$ matrix with entries in R. In this way, we obtain a linear representation of the braid group B_n in the linear group $\mathrm{GL}_n(R)$. A typical case is when R is the ring of all Laurent polynomials $\mathbb{Z}[t, t^{-1}]$, in which case the representation is the Burau representation.

Definition. We say that an LD-system $(S, *)$ is *orderable* if there is a linear ordering $<$ on S satisfying $x < x * y$ for all x, y.

Note that all the LD-systems listed so far are certainly *not* orderable. Indeed, all satisfy the *idempotency law*, *i.e.*, $x * x = x$ always holds, whereas, in an orderable LD-system, we must have in particular $x < x * x \neq x$ for each x. However, the following result was established in 1991.

Theorem 7. [7, 8] *There exist orderable LD-systems (namely free LD-systems).*

1.5. **From Theorem 7 to Theorem 2.** We claim that Theorem 2 (existence of the braid ordering) is a direct and natural consequence of Theorem 7. It is not hard to see that Theorem 2 relies on two main results.

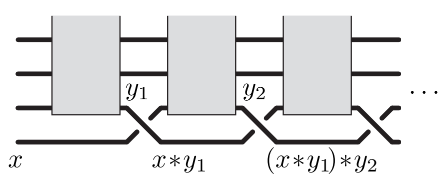
Proposition 8. *A σ-positive braid word is never trivial.*

Proposition 8 implies that the relation $<$ involved in Theorem 2 is actually an ordering,

Proposition 9. *This ordering is linear, i.e., any two braids are comparable.*

Once the existence of an orderable LD-system is granted, it is easy to understand why Proposition 8 and 9 are true.

Sketch of proof of Proposition 8. Assume that $(S, *)$ is an orderable LD-system, and that w is a σ-positive braid diagram. We wish to prove that w is not trivial. To this end, we use colors from S. Then the diagram looks as follows—here we assume, without real loss of generality, that the bottom crossings are σ_1's:

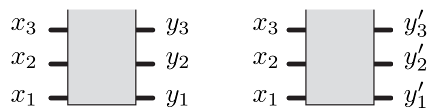


It follows from the definition of an orderable LD-system that we have

$$x < x * y_1 < (x * y_1) * y2 < ...,$$

i.e., the colors on the bottom strand keep increasing. Hence the final bottom color cannot be the initial bottom color and, therefore, the braid diagram cannot be trivial. □

Sketch of proof of Proposition 9. Assume that $(S, *)$ is again an orderable LD-system and w, w' are two braid diagrams that we wish to compare. Then we put the same initial colors $x_1, x_2, ...$ from S in both diagrams. If $(y_1, y_2, ...)$ and $(y'_1, y'_2, ...)$ are the corresponding final colors, then we can use the order of S to lexicographically compare the two sequences $(y_1, y_2, ...)$ and $(y'_1, y'_2, ...)$:



Then one of the two sequences is certainly less than the other, unless they are equal, which can happen only if β and β' are equal. □

Remark. The above sketches are not formally correct. In the case of Proposition 8, we hid the question of coloring the negative crossings σ_i^{-1}. If one wishes that every such crossing may be colored, then one has to require that the left translations of the color set are bijective—thus the latter LD-system is a rack in the sense of [13]—which actually discards orderable LD-systems. However, it can be shown with some work that, even if the latter case, one obtains well defined partial colorings, this meaning that every initial sequence of colors need not be eligible for coloring a given braid diagram but, for each such diagram, there always exist eligible sequences of colors. So the philosophy of the above sketch of proof is correct. Similarly, in the case of Proposition 9, we correctly argued that one obtains a linear ordering of braids, but we did not justify the fact that the resulting ordering coincides with the one of Theorem 2. This again requires some additional developments about orderable LD-systems, but the result is true and our sketch of proof correctly reflects the global situation.

So the reader should agree that the orderability of braid groups naturally comes from the existence of orderable LD-systems. But then the following question arises:

Question 10. *Why care about the existence of orderable LD-systems?*

1.6. **Large cardinal axioms.** The answer to Question 10 is: Because Set Theory told us.

Set theory is the mathematical study of infinity. By Gödel's incompleteness theorem, every axiomatic system for sets, in particular the standard Zermelo-Fraenkel system ZF, is incomplete, *i.e.*, there are statements that cannot be proved or disproved by that system. Then the question naturally arises of finding additional axioms that improve our description and understanding of sets (and infinity). Over the years, a consensus has emerged in the Set Theory community about the opportunity to consider a family of axioms that assert the existence of larger and larger infinities, according to a suggestion of Gödel known as Gödel's program. These axioms asserting the existence of "hyper-large" sets are usually called *large cardinal axioms.*

It is standard that a set X is infinite if and only if there exists an injection j of X into itself that is not surjective. A typical "hyper-infinite" set—the usual word is *self-similar*—is a set X such that there exists an injection j of X into itself that is not surjective and, in addition, j is a homomorphism with respect to every notion that can be defined from the membership relation $\in$. Such an injection is called a (non-trivial) *elementary embedding* of X.

For instance, the set $\mathbb{N}$ of all natural numbers is infinite, as the shift mapping $j : n \mapsto n + 1$ is a non-surjective injection of $\mathbb{N}$ into itself. But this injection is not an elementary embedding, since it is not a homomorphism with respect to the addition of natural numbers and, as explained in textbooks of Set Theory, the latter is definable from the membership relation. Actually, it is easy to check that the only injection of $\mathbb{N}$ into itself that preserves $+$, and $<$, both definable from $\in$, is the identity mapping, which shows that $\mathbb{N}$ is not self-similar. In fact, any self-similar set has to be really huge.

1.7. **Ranks.** Here comes the most bizarre and counter-intuitive notion.

Definition. A *rank* is a set R such that every function of R into itself is an element of R.

In the way the above definition is stated, it is not clear that nonempty ranks exist. Actually, the technical definition is not exactly the one above, but, once again, we are not cheating the reader and the spirit of the statement is correct.

Now, let us mix the two notions of a self-similar set and of a rank. So assume that R is a self-similar rank, and that i, j are elementary embeddings of R. By hypothesis, i applies to each element of R. As j is a mapping of R to R, then, by the defining property of a rank, j belongs to R and, therefore, i applies to j, thus yielding a new object $i(j)$. Then, j is a function, being a function is definable in Set Theory, and i preserves every property that is definable in Set Theory, so $i(j)$ is again a function. Moreover, being an elementary embedding turns out to be also definable in Set Theory, hence $i(j)$ is not only a function, but even an elementary embedding. In other words, the mapping

$$i, j \mapsto i(j)$$

defines a binary operation on the family of all elementary embeddings of R.

Next, assume that i, j, k, ℓ are elementary embeddings of R, and we have $\ell = j(k)$. "Being the image under" is definable in Set Theory, hence, as i is an elementary embedding, ℓ being is the image of k under j implies that $i(\ell)$ is the image of $i(k)$ under $i(j)$, *i.e.*, $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$. In other words, we have

$$i(j(k)) = i(j)(i(k)).$$

This means that the application operation on elementary embedding satifies the LD law, and that the family of all elementary embeddings of R equipped with the application operation is an LD-system.

In particular, if we start with one elementary embedding j of a self-similar rank, we can consider the closure of $\{j\}$ under the application operation, *i.e.*, the smallest family that contains j and is closed under the application operation. This family consists of applying j to itself in all possible ways, iteratively. Its elements are called the *iterates* of j, and the family of all iterates of j is denoted $\mathrm{Iter}(j)$. Typical iterates of j are $j(j)$, $j(j(j))$, $j(j)(j)$, etc. The previous discussion implies

Proposition 11. *If j is an elementary embedding of a self-similar rank, then* $\mathrm{Iter}(j)$ *equipped with the application operation is an LD-system.*

1.8. **A bizarre situation.** Let us recall that our purpose is to explain why some attention had been given to the question of the existence of orderable LD-systems. The answer lies in the conjunction of two results, established independently and almost simultaneously in 1989.

Theorem 12. [6] *If there exists at least one orderable LD-system, then the word problem of the LD law is algorithmicatlly solvable, i.e., there exists an algorithm that, starting with two formal terms t, t', decides whether t and t' become equal when the LD law is assumed.*

Theorem 13 (Laver, [22])**.** *If j is an elementary embedding of a self-similar rank, then* $\mathrm{Iter}(j)$ *is an orderable LD-system.*

A consequence of these results is that, *if there exists a self-similar rank*, then the word problem of the LD law is solvable. This created a very strange situation. Indeed, the existence of a self-similar rank is a large cardinal axiom, thus an unprovable strong logical assumption, whereas the word problem of the LD law is an algorithmic question that only involves finite objects, and there exists no visible connection between the set theoretical assumption and a problem of algorithmic combinatorics.

The reasons for looking for an orderable LD-system should now be clear: it was natural to look for such an algebraic object because it was needed to establish the decidability of the word problem of the LD law without appealing to an unprovable set theoretical axiom. That was done in [8] by introducing a certain group that describes the geometry of the LD law in some sense, and which is an analogue of Thomson's groups F and V—see [9] for more details. As this group turns out to be an extension of Artin's braid group B_∞, the braid applications, in particular the existence of the D-order, came as a bonus.

1.9. **Is the braid order an application of Set Theory?** We have nearly completed our journey and explained the connection between sets and braids. The last question to raise in view of Theorems 12 and 13 is: Why care about the LD-systems $\mathrm{Iter}(j)$? The answer to this last question is precisely Theorem 4. We shall not explain what Π^1_1-determinacy means, but it is enough to say here that this is a strong set-theoretical property. So Theorem 4 clearly showed that the LD-systems $\mathrm{Iter}(j)$, having strong consequences, must be non-trivial structures worth a closer investigation—other results in this direction were also established in [21].

We thus now obtained a continuous path from Theorem 4, a result about sets, to Theorem 2, a result about braids.

As a last question, we can wonder whether the braid order described in Theorem 2 is or not an application of Set Theory. Formally, it is not: braids appeared when sets disappeared, and they appeared precisely in order to avoid using sets. But one may also argue that, in essence, the answer is positive. Orderable LD-systems have been investigated because Set Theory showed they might exist and be involved in deep phenomena, and one may doubt that such algebraic systems would have been investigated without the motivation provided by Set Theory.

To conclude this part, let us propose an analogy with a situation that is rather common with physics, namely using physical assumptions to guess some statement that is subsequently passed to the mathematician for a formal rigorous proof. The situation here is quite similar: using logical assumptions (here the existence of a self-similar rank), one guesses some statement (here the existence of an orderable

LD-system), and then passes it to the mathematician for a formal rigorous proof, *i.e.*, a proof that involves no extra logical axiom.

2. Connection with knots

We turn to the second part, namely the recently discovered connections between the D-ordering of braids and knots and links. We shall mention two lines of research. The first one, initiated by Malyutin and Netsvetaev, and recently developed by Matsuda and Ito, consists in showing that, if a braid β is very small or very large in the braid ordering, then the properties of its closure $\widehat{\beta}$ can be read easily. So, the family of knots and links that are closure of very small or very large braids makes an interesting family for further investigation. The second approach, which is still at a very preliminary step, consists in using some recently introduced normal forms on braid monoids to investigate the braid conjugacy relation and, conjecturally, Markov equivalence.

2.1. **The floor.** As well as any linear ordering, the D-ordering makes the braid group B_n into a line. It is not hard to see that the powers of the central element Δ_n^2 are unbounded in $(B_n, <)$ and, therefore, they partition B_n into intervals.

Definition. (See Figure 5.) For β in B_n, the *floor* $\lfloor\beta\rfloor$ is the unique integer m for which we have $\Delta_n^{2m} \leqslant \beta < \Delta_n^{2m+2}$.

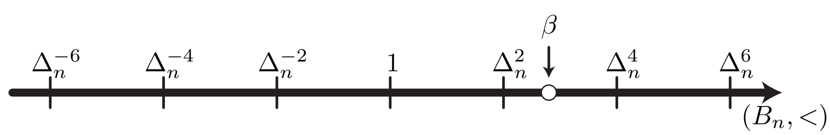


Figure 5. The powers of Δ_n^2 partition B_n into a sequence of intervals. The floor of a braid β specifies to which interval β belongs: for instance, the floor of β above is 1

It follows from the properties of the D-ordering and of those of Δ_n^2 that the floor is nearly a homomorphism of the group B_n to the group of integers.

Proposition 14 (Malyutin–Netsvetaev, [25])**.** *Define the* stable floor $\lfloor\beta\rfloor_s$ *of a braid* β *to be* $\lim_p\lfloor\beta^p\rfloor/p$. *Then the stable floor is a pseudo-character on* B_n *with defect* 1, *i.e., for all braids* β, β', *one has* $\lfloor\beta^p\rfloor_s = p\ \lfloor\beta\rfloor_s$, *and*

$$\left|\lfloor\beta\beta'\rfloor_s - \lfloor\beta\rfloor_s - \lfloor\beta'\rfloor_s\right| \leqslant 1$$

Actually the stable floor is the only pseudo-character on B_n that is positive on braids larger than 1 in the D-ordering and is 1 on Δ_n^2. It is known that the space of pseudo-characters on B_n is infinite-dimensional, but very few concrete examples are known, except the exponent sum (which has zero defect, *i.e.*, that is a homomorphism) and the signature (which, according to Gambaudo–Ghys [17] has defect n).

2.2. **Very small and very large braids.** As was said above, the general philosophy is that, when a braid is very small or very large in the D-ordering, *i.e.*, if the absolute value of the floor is large enough, then the properties of the closure are easily readable from the braid.

The point of the argument is that, for every template move M, there exists a number r_M such that, if the absolute value of the floor of a braid β is larger than r_M, then the closure of β is not eligible for M.

Theorem 15 (Malyutin–Netsvetaev, [25])**.** *If an n-strand braid β satisfies $\beta < \Delta_n^{-4}$ or $\beta > \Delta_n^4$, then the link $\widehat{\beta}$ is prime and non-trivial.*

Idea of the proof. If χ a pseudo-character on B_n with defect d whose restriction to B_{n-1} is zero, then $|\chi(\beta)| > d$ implies that $\widehat{\beta}$ is not eligible for the exchange move, whereas the Birman–Menasco theory implies that a non-prime knot can be transformed into a composite knot by a sequence of exchange moves. Applying this to the stable floor gives the result. □

Theorem 16 (Malyutin–Netsvetaev, [25])**.** *For each n, there exists $r(n)$ such that, if an n-strand braid β satisfies $\beta < \Delta_n^{-2r(n)}$ or $\beta > \Delta_n^{2r(n)}$, then the link $\widehat{\beta}$ is represented by a unique conjugacy class in B_n, i.e., $\widehat{\beta}'$ can be isotopic to $\widehat{\beta}$ only if β' is conjugated to β.*

Idea of the proof. Again by the Birman-Menasco Markov Theorem Without Stabilization, there exist finitely template moves for each fixed n, so the supremum of the corresponding integers r_M is finite. When the absolute value of the floor is larger than this supremum, the only way to transform the braid is to apply conjugacy. □

It is proved in [25] that $r(3) \leqslant 3$ holds. Then Matsuda (personal communication) announced $r(4) \leqslant 4$, Ito announced $r(3) = 2$ in [18]—which is optimal—and he conjectured that $r(n) \leqslant n-1$ might hold for each n. More recently, Ito (personal communication) announced a proof of the inequality $r(n) \leqslant n+1$ for every n.

In the same vein, Ito announced (personal communication) a proof of the following statement: if β is an n-strand braid satisfying $\beta < \Delta_n^{-2k-2}$ or $\beta > \Delta_n^{2k+2}$, then the braid index of $\widehat{\beta}$ is at least the minimum of k and n. The reason is that $\widehat{\beta}$ cannot be eligible for any template move connecting the closure of an n-strand braid and that of an k-strand braid.

Finally, we mention two more results illustrating the principle that every braid β that is very small or very large in the D-ordering admits a closure whose properties can be read from those of β easily. The first one involves the genus of the closure.

Theorem 17 (Ito, [19])**.** *If an n-strand braid β satisfies $\beta < \Delta_n^{-2k-2}$ or $\beta > \Delta_n^{2k+2}$, then the genus of $\widehat{\beta}$ is larger than $(k(n+2)-2)/4$.*

The second result involves the Nielsen–Thurston classification which takes a very simple form for very small and very large braids.

Theorem 18 (Ito, [19])**.** *If an n-strand braid β satisfies $\beta \leqslant \Delta_n^{-4}$ or $\beta \geqslant \Delta_n^4$ and $\widehat{\beta}$ is a knot, then β is periodic (resp. reducible, resp. pseudo-Anosov) iff $\widehat{\beta}$ is a torus knot (resp. a satellite knot, resp. a hyperbolic knot).*

The above simple equivalence fails in general: for instance, the trefoil knot (a torus knot) is the closure of σ_1^3 (a periodic braid), $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ (a reducible braid), and $\sigma_1^3\sigma_2^{-1}$ (a pseudo-Anosov braid).

2.3. **Two functions.** We now come to aother, much more speculative approach relying on the following deep result about the D-ordering.

Theorem 19 (Laver, [23])**.** *For every braid β and every i, one has $\beta^{-1}\sigma_i\beta > 1$.*

This property, called the *Subword Property*, implies in particular that, for each n, the restriction of the D-order to the monoid B_n^+ (the submonoid of B_n^+ generated by $\sigma_1, ..., \sigma_{n-1}$) is a well-order, *i.e.*, every nonempty subset admits a (unique) minimal element. It is then natural to introduce the following functions.

Definition. For β in B_n^+, put

$$\mu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ is conjugate to } \beta\}, \tag{20}$$

$$\nu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ is Markov equivalent to } \beta\}. \tag{21}$$

Any method for computing the function μ (*resp.* ν) would lead to a solution to the braid conjugacy problem (*resp.* the link isotopy problem). However, this approach is of little use as long as no such practical method is known.

2.4. **The alternating normal form.** The D-ordering of braids is a complicated object. For instance, this order is not Archimedian, *i.e.*, there exist β, β' larger than 1 satisfying $\beta^p < \beta'$ for each p, and it is not even Conradian, *i.e.*, there exist braids β, β' larger than 1 satisfying $\beta < \beta'\beta^p$ for each p. Therefore, whatever this exactly means, controlling the order effectively is not so easy. For instance, there is no simple connection between the order and the standard normal form of braids, namely the so-called greedy normal form(s) associated with the Garside structure(s) of braid groups [9, 12].

However, the good news are the recent introduction of two new normal forms, called the alternating and the rotating normal form, for braids. Contrary to the greedy normal form, these normal forms admit a simple connection with the D-order, bringing a reasonable hope to compute the functions μ and ν in the future.

The principle underlying the alternating normal form is the observation that, for each braid β in the monoid B_n^+ of positive n-strand braids, there exists a unique finite sequence $(..., \beta_2, \beta_1)$ of braids in B_{n-1}^+, called the *Φ_n-splitting of β*, such that, using Φ_n for the n-flip automorphism of B_n that exchanges σ_i and σ_{n-i} for each i, one has

$$\beta = ... \cdot \Phi_n(\beta_4) \cdot \beta_3 \cdot \Phi_n(\beta_2) \cdot \beta_1$$

and, for each k, the braid β_k is the maximal right-divisor of $... \cdot \Phi_n(\beta_{k+1}) \cdot \beta_k$ that lies in B_{n-1}^+—see Figure 6.

By iterating the splitting process, one easily obtains a unique expression, called the alternating normal form, for each braid of B_n^+. For our purpose, the important point is that the alternating normal form is connected with the D-order in a simple way.

Theorem 22 ([10], building on [4])**.** *The D-order of B_n^+ is the ShortLex-extension of the D-order of B_{n-1}^+ associated with the Φ_n-splitting: for β, β' in B_n^+, the relation $\beta < \beta'$ holds if and only if the Φ_n-splitting of β is shorter than that of β' or they have the same length and the splitting of β is lexicographically smaller.*

Remark. For Theorem 22 to be readily true, one has to replace the D-ordering with its Δ_n-conjugated version where one compares $\Phi_n(\beta)$ and $\Phi_n(\beta')$ instead of β and β'. In the latter ordering, σ_1 is smaller than σ_n, whereas, in the original D-order, σ_1 is larger than σ_n. Of course, the two orders are essentially equivalent.

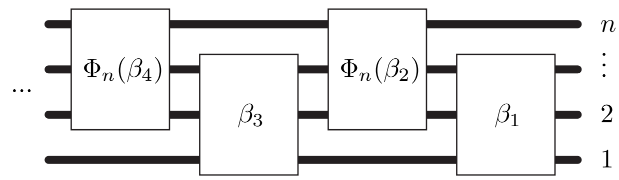


FIGURE 6. The Φ_n-splitting of a braid of B_n^+: a distinguished decomposition of every positive n-strand braid into a finite sequence of $(n-1)$-strand braids. One extracts the maximal right fragment that involves the strands 1 to $n-1$, and 2 to n, alternatively

2.5. **The rotating normal form.** The rotating normal form is analogous to the alternating normal form but, instead of involving the submonoid B_n^+ of B_n generated by the Artin generators σ_i, it appeals to the so-called dual braid monoid B_n^{+*} generated by the Birman–Ko–Lee generators of [2].

Definition (Birman–Ko–Lee, [2])**.** The *dual* braid monoid B_n^{+*} is the submonoid of the braid group B_n generated by the elements $(a_{i,j})_{1 \leqslant i < j \leqslant n}$ with

$$a_{i,j} = \sigma_{j-1}^{-1} ... \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} ... \sigma_{j-1}.$$

The braid $a_{i,j}$ corresponds to a diagram where the jth strand crosses over the ith strand behind all intermediate strands. It is known that the monoid B_n^{+*} admits the same sort of algebraic structure as the monois B_n^+, namely what is now called a *Garside structure*. Then, there exists for B_n^{+*} an analogue of the Φ_n-splitting and the alternating normal form. In the current case, the flip automorphism Φ_n has to be replaced with the so-called rotating automorphism ϕ_n that maps $a_{i,j}$ to $a_{i+1,j+1}$, where indices are taken mod. n. Provided braids are drawn on a cylinder rather than on a rectangle, ϕ_n corresponds to a $(2\pi/n)$-rotation, and the ϕ_n-splitting corresponds to the scheme of Figure 7.

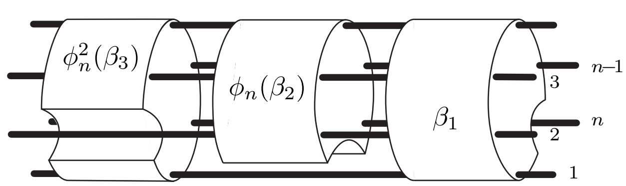


FIGURE 7. The ϕ_n-splitting of a braid of B_n^{+*}: another distinguished decomposition into a finite sequence of $(n-1)$-strand braids

In this way, one obtains a normal form on B_n^{+*}. Once again, the nice point is the following connection with the D-order.

Theorem 23 (Fromentin, [14, 15])**.** *The D-order of B_n^{+*} is the ShortLex-extension of the D-order of B_{n-1}^{+*} associated with the ϕ_n-splitting.*

Theorem 23 is more than a counterpart of Theorem 22, because the rotating normal form has some nice combinatorial properties that the alternating normal does not share. Due to the highly redundant character of the Birman–Ko–Lee generators, selecting a distinguished expression with respect to these generators is more difficult than doing it with respect to the Artin generators, and, therefore, it is not surprising that the obtained normal form turns out to be a more powerful tool.

2.6. **A conjecture.** With the alternating and rotating normal forms of braids, we now have practical ways of controlling the braid order, making it reasonable to hope for explicit computational formulas expressing parameters connected with the D-ordering in terms of the normal forms. For instance, a typical result in this direction is that, for β in B_n^+, the floor of β defined in Section 2.1 is twice the length of the Φ_n-splitting of β diminished by 2 (except for very small values of the latter).

The computation of the μ-function of Section 2.3 has not yet been completed, but it seems now accessible. Let us mention the following simple formula, that has been checked experimentally for a large number of braids.

Conjecture 24 (D., Fromentin, Gebhardt)**.** *For β in B_3^+, one has*

$$\mu(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1 \cdot \mu(\beta) \cdot \sigma_1^2.$$

Establishing this formula and various similar computational rules should lead to the practical computation of the μ function on B_3^+, and subsequently on B_n^+. When this is done, addressing similar questions for the ν function (the one where conjugacy is replaced with Markov equivalence) might become a reasonable goal.

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