GEOMETRIC PRESENTATIONS FOR THOMPSON'S GROUPS

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Abstract. Starting from the observation that Thompson's groups F and V are the geometry groups respectively of associativity, and of associativity together with commutativity, we deduce new presentations of these groups. These presentations naturally lead to introducing a new subgroup $\mathfrak{S}_\bullet$ of V and a torsion free extension $B_\bullet$ of $\mathfrak{S}_\bullet$. We prove that $\mathfrak{S}_\bullet$ and $B_\bullet$ are the geometry groups of associativity together with the law $x(yz) = y(xz)$, and of associativity together with a twisted version of this law involving self-distributivity, respectively.

Previous work showed that associating to an algebraic law a so-called geometry group that captures some specific geometrical features gives useful information about that law: the approach proved instrumental for studying exotic laws like self-distributivity $x(yz) = (xy)(xz)$ [7] or $x(yz) = (xy)(yz)$ [8]. In the case of associativity [6], the geometry group turns out to be Thompson's group F, not a surprise as the connection of the latter with associativity has been known for long time [20].

In this paper, we develop a rather general method for constructing geometry groups and, chiefly, finding presentations for these groups, and we apply this method in the case of associativity—thus finding presentations of F—and of associativity plus commutativity, thus finding new presentations of Thompson's group V, as the latter happens to be the involved geometry group.

In the case of F, the new presentation, which is centered around MacLane's pentagon relation, is more symmetric than the usual ones and it leads to an interesting lattice structure connected with Stasheff's associahedra; this structure will be investigated in [10]. In the case of V, on which we concentrate here, we describe several new presentations corresponding to various choices of the generators. In each case, once some preliminary combinatorial results are established, proving that a candidate list of relations actually makes a presentation is a straightforward application of our general method and a very simple argument.

Perhaps the main merit of the above presentations of V is to naturally lead to introducing two new groups which seem interesting in themselves. Indeed, one of these presentations explicitly includes the Coxeter presentation of the symmetric group $\mathfrak{S}_\infty$ (direct limit of the $\mathfrak{S}_n$'s), thus emphasizing the existence of a copy of $\mathfrak{S}_\infty$ inside V. When we extract those generators and relations that correspond to F and to that copy of $\mathfrak{S}_\infty$, we obtain a subgroup $\mathfrak{S}_\bullet$ of V, and, when we remove the torsion relations $s_i^2 = 1$ in the involved Coxeter presentation, we obtain an extension $B_\bullet$ of $\mathfrak{S}_\bullet$: the connection between $B_\bullet$ and $\mathfrak{S}_\bullet$ is the same as the one between Artin's braid group B_∞ and $\mathfrak{S}_\infty$.

The algebraic and geometric properties of the groups $\mathfrak{S}_\bullet$ and, specially, $B_\bullet$ are very rich. In the current paper, we address these groups only from the viewpoint of geometry groups, and we prove two results: on the one hand, the group $\mathfrak{S}_\bullet$ is itself a geometry group, namely that of associativity together with the left semi-commutativity law $x(yz) = y(xz)$; on the other hand, in some convenient sense, $B_\bullet$ is the geometry group for associativity together with a twisted version of semi-commutativity in which $x(yz) = y(xz)$ is weakened into $x(yz) = x[y](xz)$, where $x, y \mapsto x[y]$ is a second binary operation obeying a self-distributivity condition.

The groups $\mathfrak{S}_\bullet$ and $B_\bullet$ to which our approach leads turn out to be (isomorphic to) the groups $\widehat{V}$ and $\widehat{BV}$ recently introduced and investigated by M. Brin in [1, 2, 3]. The current

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work can be seen as an independent rediscovery of these groups. Let us mention still another approach to $B_\bullet$ as a group of so-called parenthesized braids: see [9], which contains a thorough study of $B_\bullet$. Various groups connecting Thompson's groups and braids, some of them close to $B_\bullet$, also appear in [14, 12, 16].

The paper is organized as follows. In Section 1, we describe in a general context the method that is used several times in the paper for identifying a presentation of a group. In Section 2, we investigate the (easy) case of associativity and Thompson's group F as a warm-up. In Section 3, we address the more interesting case of associativity together with commutativity, and obtain in this way several new presentations of V. In Section 4, we consider the case of semi-commutativity, and of the corresponding group $\mathfrak{S}_\bullet$. Finally, Section 5 is devoted to the group $B_\bullet$ and its connection with twisted semi-commutativity and self-distributive operations—in this section, some algebraic results about $B_\bullet$ are borrowed from [9].

Remark on notation. This paper involves both Thompson's groups and braid groups. Different notational conventions exist. As our approach is mainly oriented toward the group $B_\bullet$, and also for the reasons listed in [4], we chose the braid conventions, hence using actions on the right—hence xy means "x then y"—and numbering the generators from 1. To avoid confusion, we use a specific notation, namely a_i, for the generators of F, so that our a_i corresponds to the standard generator x_{i-1}^{-1} or X_{i-1}^{-1} of [5].

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1. A method for finding presentations

Throughout the paper, $\mathbf{N}$ denotes the set of all positive integers (0 excluded).

In the sequel, we address the problem of finding a presentation of a group several times, and we solve it using the same argument. So it makes sense to describe this common method first. Although perhaps never described explicitly, the latter was already used in [7].

1.1. **Partial group actions.** The situation we investigate is essentially that of a group action. However, our framework is both weaker and stronger than the standard one. The weakening is that the actions we consider are partial in that every element of the group need not act on every element; the strengthening is that our actions satisfy a strong freeness hypothesis, namely the existence of elements with a trivial stabilizer.

Several weak forms of group action may be thought of. The one convenient here is as follows. It is essentially equivalent to the one investigated in [17] (in the case of groups)—see also [18]—and in [21] (in the case of monoids).

Definition. Let G be a group, or a monoid. We define a *partial (right) action* of G on a set T to be a mapping ϕ of G into the partial injections of T into itself such that, writing $t \bullet g$ for the image of t under $\phi(g)$, the following conditions are satisfied:

(PA_1) For every t in T, we have $t \bullet 1 = t$;

(PA_2) For all g, h in G and t in T, if $t \bullet g$ is defined, then $(t \bullet g) \bullet h$ is defined if and only if $t \bullet gh$ is, and, in this case, they are equal;

(PA_3) For each finite family $g_1, \dots, g_n$ in G, there exists at least one element t in T such that $t \bullet g_1, \dots, t \bullet g_n$ are defined.

Note that, in the case of a partial action, $t \bullet gh$ being defined does not guarantee that $t \bullet g$ is. However, the following is easy:

Lemma 1.1. *Assume that ϕ is a partial action of a group G on a set T. Then*

(i) The relations $t' = t \bullet g$ and $t = t' \bullet g^{-1}$ are equivalent;

(ii) The relation $(\exists g \in G)(t' = t \bullet g)$ is an equivalence relation on T.

(iii) The stabilizer of each element of T is a subgroup of G.

Proof. As gg^{-1} is 1, (PA_2) implies that, if $t \bullet g$ is defined, then $(t \bullet g) \bullet g^{-1}$ is defined if and only if $t \bullet 1$ is, which is true by (PA_1). Then we find $(t \bullet g) \bullet g^{-1} = t \bullet 1 = t$. Hence the relation of (ii) is symmetric; (PA_1) implies that it is reflexive, and (PA_2) that it is transitive. Finally, by (PA_2) and (i), $t \bullet g = t \bullet g = t$ implies that $t \bullet (gg')$ and $t \bullet g^{-1}$ are defined and equal t. □

Thus, like an ordinary (total) action, a partial action of a group on a set T defines a partition of T into disjoint orbits. In the sequel we often use presentations and expressions of the elements of a group by words. We fix the following notation.

Definition. Assume that G is a group and that X is a subset of G. We denote by $W(X)$ the set of all words built using letters from $X \cup X^{-1}$, *i.e.*, all finite sequence of such letters. For w in $W(X)$, we usually denote by $\overline{w}$ the evaluation of w in G.

In the case of a partial group action, it will be convenient to extend the action to words:

Definition. Assume that G is a group with a partial action on T, and X is a subset of G. For t in T and w in $W(X)$, we define $t \underline{\bullet} w$ to be $t \bullet \overline{w}$ whenever $t \bullet \overline{w_0}$ is defined for each prefix w_0 of w, and to be undefined otherwise.

Note that different words representing the same element of the group may act differently: for instance, for every x in X, the word xx^{-1} and the empty word ε represent 1 in G, but, for t in T, the we always have $t \underline{\bullet} \varepsilon = t$, while $t \underline{\bullet} xx^{-1} = t$ is true only if $t \bullet x$ is defined. However, applying (PA_3) to the (finite) family consisting of all prefixes of w gives

Lemma 1.2. *Assume that the group G has a partial action on T and X is a subset of G. Then, for each word w in $W(X)$, there exists at least one element t of T such that $t \underline{\bullet} w$ is defined.*

1.2. **An injectivity criterion.** Our criterion for recognizing presentations is based on the following easy remark.

Proposition 1.3. *Let $\pi : \widetilde{G} \to G$ be a surjective group homomorphism. Assume that G has a partial action on T and there exists a map $f : T \to \widetilde{G}$ such that*

$$f(t \bullet \pi(x)) = f(t) \cdot x. \tag{1.1}$$

holds for every x in some set that generates $\widetilde{G}$ and every t in T such that $t \bullet \pi(x)$ exists. Then π is an isomorphism.

Proof. Let X be the involved generating set of $\widetilde{G}$. First, for every w in $W(X)$, we have

$$f(t \underline{\bullet} \pi(w)) = f(t) \cdot \overline{w}, \tag{1.2}$$

where $\pi(w)$ is the word obtained by replacing each letter x in w with $\pi(w)$. We prove this using induction on the length ℓ of w. For $\ell = 1$ and w consisting of one letter in X, (1.2) is true by hypothesis. Assume that w consists of one letter in X^{-1}, say $w = x^{-1}$. By Lemma 1.1, $t' = t \bullet \pi(x)^{-1}$ is equivalent to $t = t' \bullet \pi(x)$. Hence, if $t \bullet \pi(x)^{-1}$ exists, so does $(t \bullet \pi(x)^{-1}) \bullet \pi(x)$, and (1.1) gives

$$f(t) = f((t \bullet \pi(x)^{-1}) \bullet \pi(x)) = f(t \bullet \pi(x)^{-1}) \cdot x,$$

hence $f(t \bullet \pi(x)^{-1}) = f(t) \cdot x^{-1}$. Assume now $w = w_1 w_2$, with w_1, w_2 shorter than w. By definition, $t \underline{\bullet} \pi(w)$ being defined means that $t \underline{\bullet} \pi(w_1)$ and $(t \underline{\bullet} \pi(w_1)) \underline{\bullet} \pi(w_2)$ are defined, and, then, (PA_2) and the induction hypothesis give

$$f(t \underline{\bullet} \pi(w)) = f((t \underline{\bullet} \pi(w_1)) \underline{\bullet} \pi(w_2)) = f(t \underline{\bullet} \pi(w_1)) \cdot \pi(\overline{w_2}) = (f(t) \cdot \pi(\overline{w_1})) \cdot \pi(\overline{w_2}) = f(t) \cdot \pi(\overline{w}).$$

Now let g be an element of $\widetilde{G}$ satisfying $\pi(g) = 1$. Let w be a word in $W(X)$ representing g. By Lemma 1.2, there exists t in T such that $t \underline{\bullet} \pi(w)$ is defined. Then, (1.2) gives

$$f(t) = f(t \bullet \pi(g)) = f(t \underline{\bullet} \pi(w)) = f(t) \cdot \overline{w} = f(t) \cdot g,$$

hence $g = 1$. □

1.3. **Group presentations.** If G is a group and R is a list of relations satisfied in G by the elements of some generating subset X, there exists a surjective homomorphism of the group $\langle X\,;R\rangle$ onto G. Proving that $(X;R)$ is a presentation of G amounts to proving that the above morphism is injective, and this is where Proposition 1.3 can be used.

In the sequel, we shall consider partial actions that satisfy strong freeness conditions. For G acting on T and $S \subseteq T$, we denote by $S \bullet G$ the set of all $s \bullet g$ for s in S and g in G.

Definition. Assume that G has a partial action on T. A subset S of T is said to be *discriminating* if, in (PA_3), we can require $t \in S \bullet G$, no two elements of S lie in the same G-orbit, and each element in S has a trivial stabilizer.

The first condition means that there is an induced partial action on $S \bullet G$, while the other ones guarantee that, for each t in $S \bullet G$, there exists a unique s in S and a unique g in G satisfying $t = s \bullet g$. In this case, we can select words describing the connection between the elements of S and the elements of their orbits. When R is a family of relations for a group, we denote by $\equiv_R$ the associated congruence. Our criterion takes the following form.

Proposition 1.4. *Let G be a group with a partial action on a set T. Let X be a subset of G and R be a collection of relations satisfied in G by the elements of X. Assume that S is a discriminating subset of T and that, for each s in S and t in the G-orbit of s, a word w_t in $W(X)$ is chosen so that $t = s \bullet \overline{w_t}$ holds. Then a necessary and sufficient condition for $(X;R)$ to be a presentation of G is that, for all t, t' in $S \bullet G$ and x in X,*

$$t' = t \bullet x \qquad \textit{implies} \qquad w_{t'} \equiv_R w_t \cdot x. \tag{1.3}$$

Proof. We begin with an auxiliary claim, namely that $t' = t \bullet g$ implies $\overline{w_{t'}} = \overline{w_t} \cdot g$ for all t, t' in $S \bullet G$ and g in G. Indeed, assume $t' = t \bullet g$. Let s be an element of S in the orbit of t. Then s also belongs to the orbit of t', and, by hypothesis, we have $t = s \bullet \overline{w_t}$ and $t' = s \bullet \overline{w_{t'}}$. On the other hand, we also have $t' = t \bullet g = (s \bullet \overline{w_t}) \bullet g$, hence $t' = s \bullet (\overline{w_t} \cdot g)$. The hypothesis that S is discriminating then implies $\overline{w_{t'}} = \overline{w_t} \cdot g$, as expected.

Let us show that (1.3) is a necessary condition. Assume $t' = t \bullet x$. By the claim above, we deduce $\overline{w_{t'}} = \overline{w_t} \cdot x$, *i.e.*, the words $w_{t'}$ and $w_t \cdot x$ represent the same element of G. If $(X;R)$ is a presentation of G, they must be R-equivalent, and the condition is necessary.

We turn to the converse. First, let g be an arbitrary element of G. As S is discriminating, there exists t in $S \bullet G$ such that $t \bullet g$ exists. Let $t' = t \bullet g$. By the claim above, we haves $\overline{w_{t'}} = \overline{w_t} \cdot g$. Now, by hypothesis, the words $w_{t'}$ and w_t lie in $W(X)$, so their classes belong to the subgroup of G generated by X, and so does g. Hence X generates G. It remains to show that the relations of R make a presentation of G. Let $\widetilde{G}$ be the presented group $\langle X\,;R\rangle$. The set X generates G, and the hypothesis is that the relations of R are satisfied in G. Hence there exists a surjective homomorphism $\pi : \widetilde{G} \to G$ which is the identity on X, and we aim at proving that π is injective. Now, define $f : S \bullet G \to \widetilde{G}$ so that $f(t)$ is the element of $\widetilde{G}$ represented by w_t. If we assume (1.3), then $t' = t \bullet x$ implies $f(t \bullet x) = f(t) \cdot x$: this is exactly Relation (1.1) for the partial action of G on $S \bullet G$, and Proposition 1.3 then says that π must be injective. □

The previous criterion will always be used as a sufficient condition here. However knowing that the condition is also necessary guarantees that the presentations one obtains by introducing just enough relations to witness for all equivalences occurring in (1.3) are in some sense minimal. Also, adapting the criterion to the context of monoids is easy, provided the considered monoids admits left cancellation—but we shall not use this version here.

2. Thompson's group F as the geometry group of associativity

We describe now a realization of Thompson's group F as the geometry group of the associativity law. This is one way of formalizing the well-known connection between F and the associativity law, and it naturally leads to a presentation of F in terms of a family of generators indexed by binary addresses. Apart from more or less trivial geometric relations, the only relations in this presentation correspond to the well-known MacLane–Stasheff's pentagons.

2.1. **Trees and associativity.** In the sequel, we consider finite, rooted binary trees—simply called trees. The number of leaves in a tree is called its *size*. We denote by $\bullet$ the tree consisting of a single vertex and by $t_1 \cdot t_2$, or simply $t_1 t_2$, the tree with left subtree t_1 and right subtree t_2. Every tree has a unique decomposition in terms of $\bullet$ and the product.

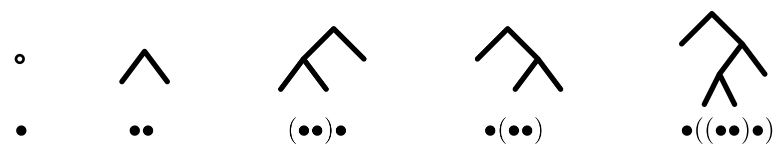


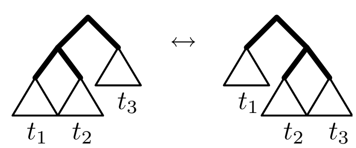
FIGURE 1. Typical trees with their decomposition in terms of $\bullet$

We also consider *L-coloured* trees, defined as trees in which the leaves wear labels—or colours—taken from the set L. We write $\bullet_x$ for $\bullet$ with label x, and T_L for the set of all L-coloured trees. We use $T_\emptyset$ for the set of all uncoloured trees, and see it as a subset of $T_{\mathbf{N}}$ by identifying an uncoloured tree with the coloured tree where all leaves are labelled 1.

The associativity law

$$(\mathcal{A}) \qquad x(yz) = (xy)z$$

gives rise to an equivalence relation on (coloured) trees: two trees t, t' are equivalent up to associativity if we can transform t into t' by iteratively replacing one subtree of the form $t_1(t_2t_3)$ with the corresponding tree $(t_1t_2)t_3$, or *vice versa*:



In order to describe this action precisely, we need an indexation for the subtrees of a tree. One solution is to describe the path from the root of the tree to the root of the considered subtree using (for instance) 0 for "forking to the left" and 1 for "forking to the right".

Definition. A finite sequence of 0's and 1's is called an *address*; the empty address is denoted ϕ. For t a (coloured) tree and α a short enough address, the *α-subtree* of t is the part of t that lies below α. The set of all α's for which the α-subtree of t exists is called the *skeleton* of t.

Formally, the α-subtree is defined by the following rules: the ϕ-subtree of t is t, and, for $\alpha = 0\beta$ (*resp.* 1β), the α-subtree of t is the β-subtree of t_1 (*resp.* t_2) when t is t_1t_2, and it is undefined in other cases. For instance, for $t = \bullet((\bullet\bullet)\bullet)$ (the rightmost example in Figure 1), the 10-subtree of t is $\bullet\bullet$, while its 01- and 111-subtrees are undefined. The skeleton of t consists of ϕ, 0, 1, 10, 100, 101, 11.

Applying associativity to a tree t consists in choosing an address α in the skeleton of t and either replacing the α-subtree of t, supposed to have the form $t_1(t_2t_3)$, by the corresponding $(t_1t_2)t_3$, or performing the inverse substitution. We can see this as applying an operator.

Definition. (*i*) We denote by A the partial operator on $T_{\mathbf{N}}$ that maps every tree of the form $t_1(t_2t_3)$ to the corresponding tree $(t_1t_2)t_3$.

(*ii*) For α an address and f a partial mapping on trees, we define the *α-shift* of f, denoted $\partial_\alpha f$, to be the partial mapping consisting in applying f to the α-subtree of its argument (when the latter exists). We write ∂ for ∂_1.

(*iii*) For α an address, we put $A_\alpha = \partial_\alpha A$. We define $\mathcal{G}(\mathcal{A})$ to be the monoid generated by all A_α's and their inverses using reversed composition.

Example 2.1. (Figure 2) Let $t = \bullet(((\bullet\bullet)\bullet)(\bullet\bullet))$. Then t lies in the domain of A, as the ϕ-subtree of t, *i.e.*, t itself, is $t_1(t_2t_3)$, with $t_1 = \bullet$, $t_2 = (\bullet\bullet)\bullet$, and $t_3\bullet\bullet$. Then the image of t under A is $(t_1t_2)t_3$, *i.e.*, $(\bullet((\bullet\bullet)\bullet))(\bullet\bullet)$. Similarly, t lies in the domain of A_1, and in the images of A_1 and of A_{10}, hence in the domains of A_1^{-1} and A_{10}^{-1}. These are the only operators $A_\alpha^{\pm 1}$ applying to t.

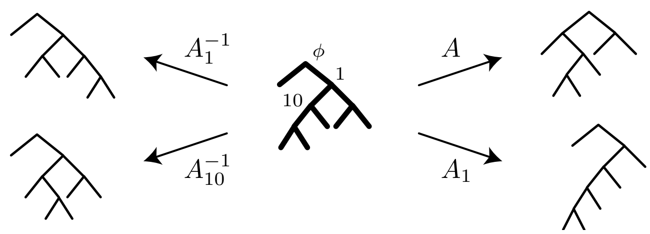


FIGURE 2. Two operators A_α and two operators A_α^{-1} apply to the tree $\bullet(((\bullet\bullet)\bullet)(\bullet\bullet))$

We thus have a partial action of the monoid $\mathcal{G}(\mathcal{A})$ on trees in the sense of Section 1; for f in $\mathcal{G}(\mathcal{A})$, we write $t \bullet f$ for the image of t under f, when it exists. We use reversed composition in $\mathcal{G}(\mathcal{A})$ so as to make our multiplication compatible with an action on the right.

By construction, two trees t, t' are equivalent up to associativity if and only if some element of $\mathcal{G}(\mathcal{A})$ maps t to t'. Thus the orbits for the partial action of the monoid $\mathcal{G}(\mathcal{A})$ are the equivalence classes with respect to associativity. In particular, there is exactly one orbit for each size inside $T_\emptyset$, and the cardinal of the orbit of size n trees is the nth Catalan number.

2.2. **Making $\mathcal{G}(\mathcal{A})$ into a group.** Except the identity mapping, the elements of $\mathcal{G}(\mathcal{A})$ are partial mappings, and the monoid $\mathcal{G}(\mathcal{A})$ is not a group, but only an inverse monoid, *i.e.*, a monoid in which, for each element g, there exists g^{-1} satisfying $gg^{-1}g = g$ and $g^{-1}gg^{-1} = g^{-1}$. For instance, the product AA^{-1} is the identity of its domain, but the latter does not contain $\bullet$.

Every inverse monoid admits a maximal quotient-group, called its universal group [15, 22]. In the general case, the universal group may be much smaller than the original monoid, typically when the latter consists of partial mappings whose domains may be disjoint. In the current case, no wild collapsing occurs, and the induced action of the universal group keeps the freeness properties of the initial monoid action. As the same construction will be used several times, we describe it in a general framework.

Definition. Two partial mappings g, g' are said *near-equal*, denoted $g \approx g'$, if there is at least one element t such that both $t \bullet g$ and $t \bullet g'$ are defined, and $t \bullet g = t \bullet g'$ holds for every such t.

Lemma 2.2. *Assume that $\mathcal{G}$ is a monoid consisting of partial self-injections of a set T that is closed under inverse, and there exists a subset S of T such that, for all $g_1, \ldots, g_n, g, g'$ in $\mathcal{G}$,*

$$\mathrm{Dom}(g_1) \cap \ldots \cap \mathrm{Dom}(g_n) \cap S \bullet \mathcal{G} \text{ is nonempty,} \tag{2.1}$$

$$g \approx g' \text{ is true whenever } t \bullet g = t \bullet g' \text{ holds for some } t \text{ in } S \bullet \mathcal{G}. \tag{2.2}$$

Then near-equality is a congruence on $\mathcal{G}$, the quotient-monoid is a group, the mappings of $\mathcal{G}$ induce a partial action of this group on T, and the set S is discriminating for this partial action.

Proof. Assume $g' \approx g'' \approx g'''$. By (2.1), there exists t in $S \bullet \mathcal{G}$ such that $t \bullet g', t \bullet g''$, and $t \bullet g'''$ are defined. Then one necessarily has $t \bullet g' = t \bullet g'''$, hence $g' \approx g'''$ by (2.2), and $\approx$ is an equivalence relation. Next, $g' \approx g''$ implies $gg' \approx gg''$ and $g'g \approx g''g$ for every g, because (2.1) guarantees that there exists t in $S \bullet \mathcal{G}$ for which $t \bullet g$, $t \bullet gg'$, $t \bullet gg''$, $t \bullet g'$, $t \bullet g'g$, $t \bullet g''$, and $t \bullet g''g$ are defined. So $\approx$ is a congruence on $\mathcal{G}$, and the quotient-monoid $\mathcal{G}/\approx$, henceforth denoted G, is well-defined. For each g in $\mathcal{G}$, we have $gg^{-1} \approx \mathrm{id}$ because $\mathrm{Dom}(g)$ is nonempty, so G is a group.

For g in $\mathcal{G}$, let us denote by $\overline{g}$ the class of g in G. For t in T, and x in G, we define $t \bullet x$ to be t' if $t \bullet g = t'$ holds for some element g of $\mathcal{G}$ satisfying $\overline{g} = x$, if such an element exists. Then

$t \bullet x$ is well-defined by definition of $\approx$, and we claim that one obtains in this way a partial action of G on T. Indeed, Condition (PA_1) is trivial. As for (PA_2), assume that $t \bullet x$ and $(t \bullet x) \bullet y$ are defined. This means that there exist g, h with $x = \overline{g}$ and $y = \overline{h}$ such that $t \bullet g$ and $(t \bullet g) \bullet h$ are defined. But, then, $t \bullet gh$ is defined, and, by construction, we have $\overline{gh} = \overline{g}\,\overline{h}$. Conversely, assume that $t \bullet x$ and $t \bullet xy$ are defined, say $t \bullet x = t'$ and $t \bullet xy = t''$. This means that there exist g, g in $\mathcal{G}$ satisfying $t \bullet g = t'$, $t \bullet g = t''$, with $\overline{g} = x$ and $\overline{g} = xy$. Let $h = g^{-1}g$. Then h belongs to $\mathcal{G}$, we have $\overline{h} = x^{-1}xy = y$, and $t' \bullet h = t''$. This shows that $(t \bullet x) \bullet y$ is defined, and equal to t''. So Condition (PA_2) is satisfied. Then (2.1) implies (PA_3) directly, and we obtain a partial action of G on T. Finally, the subset S is discriminating by (2.2). □

In order to apply the previous construction to the monoid $\mathcal{G}(\mathcal{A})$ and its action on trees, we describe the domain and the image of a generic element of $\mathcal{G}(\mathcal{A})$ explicitly.

Definition. (*i*) A mapping of $\mathbf{N}$ to $T_{\mathbf{N}}$ is called a *substitution.* If t is a tree in $T_{\mathbf{N}}$ and σ is a substitution, we denote by t^σ the tree obtained by replacing each leaf $\bullet_x$ in t with the tree $\sigma(x)$.

(*ii*) A coloured tree is said to be *injective* if its labels are pairwise distinct.

(*iii*) For g a partial mapping of $T_{\mathbf{N}}$ into itself, we say that a pair of trees (t, t') in $T_{\mathbf{N}}$ is a *seed* for g if, as a set of pairs, g is the set of all (t^σ, t'^σ) with σ a substitution.

The pair $(\bullet_1(\bullet_2\bullet_3), (\bullet_1\bullet_2)\bullet_3)$ is a seed for A: this is just saying that A consists of all pairs of the form $(t_1(t_2t_3), (t_1t_2)t_3)$. Then we have the following general result:

Lemma 2.3. *Each element of $\mathcal{G}(\mathcal{A})$ admits a seed consisting of injective trees.*

Proof. Let g be an element of $\mathcal{G}(\mathcal{A})$. We use induction on the (minimal) length of a decomposition of g in terms of the operators A_α and A_α^{-1}. The pair $(\bullet_1, \bullet_1)$ is a seed for $g = \mathrm{id}$, the pair $(\bullet_1(\bullet_2\bullet_3), (\bullet_1\bullet_2)\bullet_3)$ is a seed for $g = A$, and it is easy to define similarly a seed for $g = A_\alpha^{\pm 1}$. Otherwise, write $g = g_1g_2$. By induction hypothesis, g_1 and g_2 admit seeds, say (t_1, t'_1) and (t_2, t'_2). If t'_1 happens to coincide with t_2, then (t_1, t'_2) is a seed for g. In the general case, because t'_1 and t_2 are injective, there exist minimal substitutions σ_1 and σ_2 such that ${t'_1}^{\sigma_1}$ and $t_2^{\sigma_2}$ coincide, and, then, the pair $(t_1^{\sigma_1}, {t'_2}^{\sigma_2})$ is a seed for g. □

(Moreover, the seed is unique if the labels are requested to make an initial segment of $\mathbf{N}$.)

Corollary 2.4. *The monoid $\mathcal{G}(\mathcal{A})$ satisfies Conditions* (2.1) *and* (2.2) *of Lemma 2.2 with $T = T_{\mathbf{N}}$ and S any subset of $T_{\mathbf{N}}$ containing trees of arbitrary large size.*

Proof. Let S be a subset of $T_{\mathbf{N}}$ containing trees of arbitrary large size, and let t be an arbitrary tree. Then there exists s in S whose size is at least that of t. Using associativity, we can transform s into a tree whose skeleton includes that of t, *i.e.*, there exists g in $\mathcal{G}(\mathcal{A})$ such that $s \bullet g$ is defined and its skeleton includes that of t.

Let $g_1, \dots, g_n$ be elements of $\mathcal{G}(\mathcal{A})$, and (t_1, t'_1), ..., (t_n, t'_n) be seeds for these elements. By the above argument, there exists a tree t in $S \bullet \mathcal{G}(\mathcal{A})$ whose skeleton includes the skeletons of $t_1, \dots, t_n$, hence there exist substitutions $\sigma_1, \dots, \sigma_n$ such that $t = t_i^{\sigma_i}$ holds for each i, which implies that $t \bullet g_i$ is defined for each i. So Condition (2.1) is satisfied.

Assume that g_1, g_2 belong to $\mathcal{G}(\mathcal{A})$, and $t \bullet g_1 = t \bullet g_2$ holds for some tree t in $T_{\mathbf{N}}$. Let (t_1, t'_1), (t_2, t'_2) be seeds for g_1 and g_2 respectively. As above, there exist substitutions σ_1, σ_2 such that the trees $t_1^{\sigma_1}$ and $t_2^{\sigma_2}$ coincide, they are injective, and their common skeleton is the union of the skeletons of t_1 and t_2. The hypothesis that $t \bullet g_1$ and $t \bullet g_2$ are defined implies that the skeleton of t includes those of t_1 and t_2, hence their union. Hences there exists a substitution σ satisfying $t = (t_1^{\sigma_1})^\sigma = (t_2^{\sigma_2})^\sigma$. The hypothesis that $t \bullet g_1$ and $t \bullet g_2$ are equal then gives

$$({t'_1}^{\sigma_1})^\sigma = t \bullet g_1 = t \bullet g_2 = ({t'_2}^{\sigma_2})^\sigma.$$

This implies that the skeletons of ${t'_1}^{\sigma_1}$ and ${t'_2}^{\sigma_2}$ coincide. Moreover, the hypothesis $t_1^{\sigma_1} = t_2^{\sigma_2}$ implies that the sequence of labels in $t_1^{\sigma_1}$ and $t_2^{\sigma_2}$ coincide. As associativity does not change the order of the labels, the trees ${t'_1}^{\sigma_1}$ and ${t'_2}^{\sigma_2}$ must coincide. This means that g_1 and g_2 agree on every tree whose skeleton includes that of $t_1^{\sigma_1}$, *i.e.*, on every tree in the intersection of the domains of g_1 and g_2. In other words, $g_1 \approx g_2$ holds, and Condition (2.2) is satisfied. □

By applying Lemma 2.2, we obtain:

Proposition 2.5. *Near-equality is a congruence on the monoid $\mathcal{G}(\mathcal{A})$, and the quotient-monoid is a group. The operators $A_\alpha^{\pm 1}$ induce a partial action of this group on $T_{\mathbf{N}}$, and every subset of $T_{\mathbf{N}}$ containing trees of unbounded sizes is discriminating for this partial action.*

Definition. The *geometry group of associativity*, denoted $G(\mathcal{A})$, is defined to be the quotient-monoid $\mathcal{G}(\mathcal{A})/\approx$.

In the sequel, we still use A_α for the class of A_α in $G(\mathcal{A})$. For t a tree and g an element of $G(\mathcal{A})$, we denote by $t \bullet g$ the result of letting g act on t. The elements of $G(\mathcal{A})$ are expressed by words on $\boldsymbol{A}$, and we also use $\bullet$ for the word action, *i.e.*, we do not distinguish between $\bullet$ and $\underline{\bullet}$. But we recall that $t \bullet w$ exists only if $t \bullet \overline{w_0}$ exists for each prefix w_0 of w: for instance, $(\bullet\bullet) \bullet AA^{-1}$ is not defined, since $(\bullet\bullet) \bullet A$ is not.

It is straightforward to connect the geometry group $G(\mathcal{A})$ with Thompson's group F:

Proposition 2.6. *The group $G(\mathcal{A})$ is (isomorphic to) Thompson's group F, i.e., F is the geometry group of associativity.*

Proof. (Figure 3) We start with the definition of F as a group of orientation preserving piecewise linear homeomorphisms of the unit interval, *cf.* [5]. Let g be an arbitrary element in $\mathcal{G}(\mathcal{A})$. We map g to F as follows: let (t, t') be a seed for g; we associate with t a dyadic decomposition $0 = r_0 < r_1 < \ldots < r_n = 1$ of $[0, 1]$, and, similarly, let $0 = r'_0 < r'_1 < \ldots < r'_n = 1$ be the dyadic decomposition associated with t'; then we map g to the unique piecewise linear homeomorphism that maps r_i to r'_i and interpolates the values. We obtain in this way a morphism $\pi : \mathcal{G}(\mathcal{A}) \to F$. The homeomorphisms associated with (t, t') and (t^σ, t'^σ) coincide, and this implies that π factors through $\approx$. The injectivity of the resulting morphism follows from the fact that each element of F is determined by its values on a finite dyadic partition; its surjectivity follows from the fact that the images of A and A_1 generate F. □

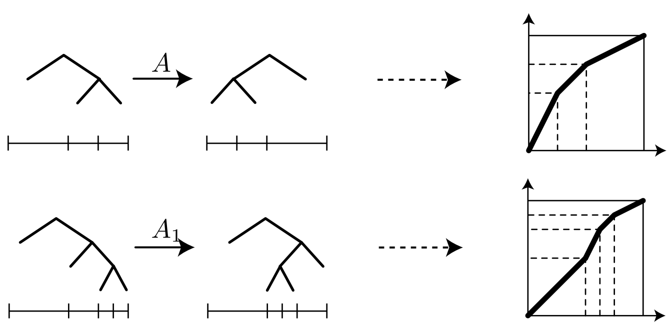


FIGURE 3. From $\mathcal{G}(\mathcal{A})$ to F: the action of A and A_1

From now on, we identify F with $G(\mathcal{A})$.

2.3. **Guessing relations in $G(\mathcal{A})$.** Considering the group F as the geometry group of associativity naturally leads to a presentation of F in terms of the generators A_α. We proceed in two steps: first, we use the geometric definition of the operators A_α to guess a list of relations; then, we prove that these relations make a presentation using the method of Section 1.

Let us look for relations between the operators A_α. We shall describe two types of relations: the *geometric* relations, and the *pentagon relations*. Geometric relations arise when we consider inheritance phenomena. Assume $t' = t \bullet A$, *i.e.*, assume that the operator A maps t to t'. Then, by definition, the 1-subtree of t' is a copy of the 11-subtree of t. It follows that performing any transformation in the latter subtree and then applying A has the same result as applying A first and performing the considered transformation in the 1-subtree of t'. Therefore, the equality

$$\partial^2 f \cdot A = A \cdot \partial f \tag{2.3}$$

holds for every (partial) mapping f on trees (Figure 4). In particular, for $f = A_\alpha$, we obtain

$$A_{11\alpha} \cdot A = A \cdot A_{1\alpha}, \tag{2.4}$$

a typical example of what we shall call a *geometric relation*.

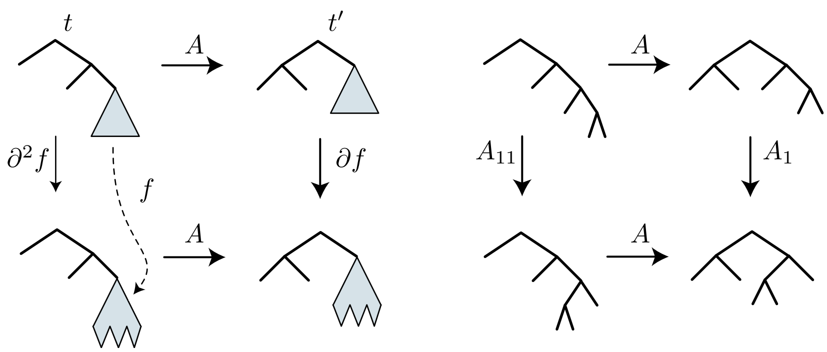


FIGURE 4. Geometric relations in $\mathcal{G}(\mathcal{A})$: the general scheme and one example

We shall say that, under the action of A, the address 1 is the *heir* of the address 11, and, more generally, that 1α is the heir of 11α. Inheritance phenomena are quite general. Under the action of A_α, for every β, the address $\alpha 00\beta$ is the heir of $\alpha 0\beta$, the address $\alpha 01\beta$ is the heir of $\alpha 10\beta$, and $\alpha 1\beta$ is the heir of $\alpha 11\beta$ under A_α. Furthermore, if we say that two addresses α, β are incompatible, denoted $\alpha \perp \beta$, if neither is of prefix of the other, *i.e.*, if there exists γ such that $\gamma 0$ is a prefix of α and $\gamma 1$ is a prefix of β, or *vice versa*, then each address β with $\beta \perp \alpha$ is its own heir under the action of A_α.

The argument leading to (2.3) gives the relation $\partial_\gamma f \cdot A_\alpha = A_\alpha \cdot \partial_{\gamma'} f$ whenever γ' is the heir of γ under A_α. In this way, we deduce a list of geometric relations in $\mathcal{G}(\mathcal{A})$, namely

$$\begin{cases} A_\beta \cdot A_\alpha = A_\alpha \cdot A_\beta \quad \text{for } \beta \perp \alpha, \\ A_{\alpha 0\beta} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 00\beta}, \\ A_{\alpha 10\beta} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 01\beta}, \\ A_{\alpha 11\beta} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 1\beta}. \end{cases} \tag{2.5}$$

The geometric relations are rather trivial, and we look for other, non-trivial relations in $\mathcal{G}(\mathcal{A})$. As can be expected, MacLane's pentagon enters the picture.

Lemma 2.7. *For each α, the following pentagon relation holds in $\mathcal{G}(\mathcal{A})$:*

$$A_\alpha \cdot A_\alpha = A_{\alpha 1} \cdot A_\alpha \cdot A_{\alpha 0}. \tag{2.6}$$

The verification (for $\alpha = \phi$) is given in Figure 5. Keeping the same name for the relations in $\mathcal{G}(\mathcal{A})$ and their counterparts in $G(\mathcal{A})$—hence in F—we can summarize the results as follows.

Definition. We denote by $\boldsymbol{A}$ the family of all A_α's, and by R_A the family of all geometry relations involving A, namely the translated copies of

$$(\square_\perp) \qquad A_{0\alpha} \cdot A_{1\beta} = A_{1\beta} \cdot A_{0\alpha},$$

$$(\square_A) \qquad A_{11\alpha} A = A A_{1\alpha}, \qquad A_{10\alpha} A = A A_{01\alpha}, \qquad A_{0\alpha} A = A A_{00\alpha};$$

plus the pentagon relations, *i.e.*, the translated copies of

$$(\pentagon) \qquad AA = A_1 A A_0.$$

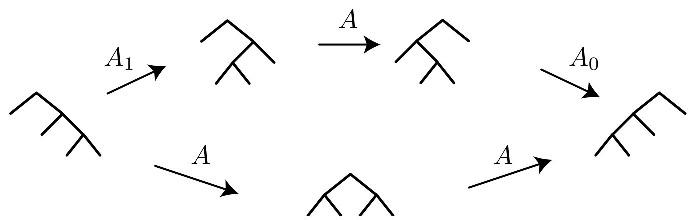


FIGURE 5. The pentagon relation

Proposition 2.8. *All relations of R_A are satisfied by the elements A_α in $G(\mathcal{A})$, i.e., in F.*

2.4. **Constructing trees.** Our next aim is to prove that the relations of Proposition 2.8 make a presentation of F. We apply the method described in Section 1, using the partial action of $G(\mathcal{A})$ on trees. We showed that any family of trees containing trees of arbitrary large size is discriminating, so, according to Proposition 1.4, two ingredients are needed, namely

- a family of trees S containing trees of unbounded size, and
- for every tree t in the orbit of S, a distinguished word w_t in $W(\boldsymbol{A})$ connecting t with some distinguished element of its orbit.

Both steps are easy: two trees are equal up to associativity if and only if they have the same size, so each family of trees containing exactly one tree of size n for each n is convenient. In the current case, we shall use the *right vines* (or combs) of Figure 6.

Definition. Let $t_1, \ldots, t_n$ be trees. We put

$$\langle t_1, \ldots, t_n \rangle = t_1(t_2 \ldots (t_{n-1} t_n) \ldots);$$

we define the *right vine* $\langle n \rangle$ to be $\langle \bullet, \ldots, \bullet \rangle$ with n times $\bullet$.

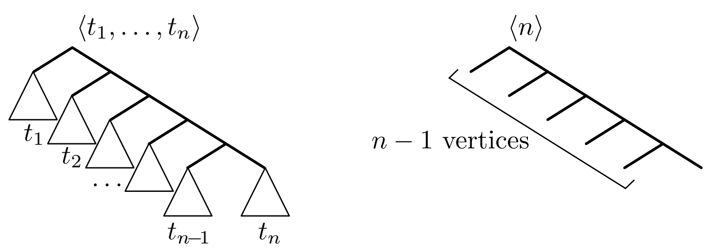


FIGURE 6. The notation $\langle t_1, \ldots, t_n \rangle$ and the right vine $\langle n \rangle$

With this notation, applying the operator A means replacing $\langle t_1, t_2, \ldots \rangle$ with $\langle t_1 t_2, \ldots \rangle$. As there are vines of each size, we immediately get:

Lemma 2.9. *Vines form a discriminating subset of $T_{\mathbf{N}}$ for the action of $G(\mathcal{A})$.*

If t is a size n tree, there exists a (unique) element of $G(\mathcal{A})$ mapping the right vine $\langle n \rangle$ to t: in order to obtain (1.3) and possibly apply Proposition 1.4, it suffices to select a distinguished word w_t representing that element, *i.e.*, to describe how t can be *constructed* from $\langle n \rangle$ using associativity. Several solutions exist. We give now an inductive definition that leads to short computations, but requires that we introduce two words w_t, w_t^* for each tree t.

Definition. (i) For w a word involving letters indexed by addresses, we denote by $\partial_\alpha w$ the word obtained by appending α at the beginning of each index; we use ∂w for $\partial_1 w$.

(ii) For each tree t, we define two words w_t, w_t^* using the inductive rules:

$$w_t = w_t^* = \varepsilon \qquad \text{for } t \text{ of size } 1,$$
$$w_t = w_{t_1}^* \cdot \partial w_{t_2}, \qquad w_t^* = w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \qquad \text{for } t = t_1 t_2.$$

The following characterization of the words w_t and w_t^* is not needed in the sequel, but it should make the construction concrete. Each tree t admits a unique decomposition in terms of the basic tree $\bullet$. Besides the algebraic notation $t_1 \cdot t_2$ for the product of t_1 and t_2, we can also use the right Polish notation in which this product is denoted $t_1 t_2 \circ$. For instance, the Polish expression of $\bullet((\bullet\bullet)\bullet)$ is $\bullet\bullet\bullet\circ\bullet\circ\circ$. In the next proposition, a length ℓ word w is considered as a sequence of symbols indexed by $\{1, \dots, \ell\}$, and $w(p)$ denotes the pth symbol in w.

Proposition 2.10. *For w a length ℓ word and $0 \leqslant p \leqslant \ell$, define the defect $\delta_w(p)$ of p in w by the rules: $\delta_w(0) = -1$, $\delta_w(p) = \delta_w(p-1) - 1$ for $w(p) = \circ$, and $\delta_w(p) = \delta_w(p-1) + 1$ otherwise. Then, for each tree t, the word w_t^* is obtained from the Polish expression of t by deleting the symbols $\bullet$, and replacing each defect i symbol $\circ$ with A_{1^i}. The word w_t is obtained similarly, except that the final symbols $\circ$, i.e., those followed by no $\bullet$, do not contribute.*

Proof. It is standard that a word w is the Polish expression of a tree if and only if the defect of each symbol is nonnegative, and the defect of the last symbol is 0. For t a tree, define the *enhanced decomposition* of t to be the Polish expression with the defect of each symbol appended. Then the enhanced decomposition of $t_1 t_2$ is the enhanced decomposition of t_1, followed by the enhanced decomposition of t_2 with all defects shifted by 1, followed by the symbol $\circ$ with 0 defect. So, the enhanced decomposition and the word w_t^* obey parallel inductive rules. Hence, as the correspondence of the proposition clearly holds for the basic tree $\bullet$, it inductively holds for every tree. A similar argument gives the connection between w_t^* and w_t. □

For instance, for $t = \bullet((\bullet\bullet)\bullet)$, the enhanced decomposition of t is $\bullet 0 \bullet 1 \bullet 2 \circ 1 \bullet 2 \circ 1 \circ 0$, and a direct translation yields $w_t^* = A_1 A_1 A$, and $w_t = A_1$ (the last two symbols $\circ$ are dismissed). A consequence of Proposition 2.10 is that, for each tree t, we have

$$w_t^* = w_t \cdot A_{1^{h-1}} \dots A_1 A, \tag{2.7}$$

where h is the length of the rightmost branch in t.

We aim at proving that the trees $\langle n \rangle$ and the words w_t satisfy the requirements of Proposition 1.4 and therefore lead to a presentation of $G(\mathcal{A})$, *i.e.*, of F. In the sequel, we use mixed expressions like $\langle p, t, q, \dots \rangle$ where p, q are numbers and t is a tree to mean $\langle \bullet, \dots, \bullet, t, \bullet, \dots, \bullet, \dots \rangle$ with p $\bullet$ in the first block and q in the second.

Lemma 2.11. *For each size n tree t and each tree t', we have*

$$\langle n \rangle \xrightarrow{w_t} t \qquad \text{and} \qquad \langle n, t' \rangle \xrightarrow{w_t^*} \langle t, t' \rangle, \tag{2.8}$$

i.e., w_t constructs t from $\langle n \rangle$, and w_t^* constructs tt' from $\langle n, t' \rangle$.

Proof. We use induction on n. For $n = 1$, the result is obvious. Otherwise, assume $t = t_1 t_2$, and let n_1 and n_2 be the respective sizes of t_1 and t_2. Then we have $w_t = w_{t_1}^* \cdot \partial w_{t_2}^*$. By induction hypothesis, $w_{t_1}^*$ maps $\langle n \rangle$, *i.e.*, $\langle n_1, n_2 \rangle$, to $t_1 \langle n_2 \rangle$, *i.e.*, $\langle t_1, n_2 \rangle$. Then, by induction hypothesis again, w_{t_2} maps $\langle n_2 \rangle$ to t_2, hence ∂w_{t_2} maps $t_1 \langle n_2 \rangle$ to $t_1 t_2$. So w_t maps $\langle n \rangle$ to $t_1 t_2$, *i.e.*, to t (Figure 7 top):

$$\langle n \rangle = \langle n_1, n_2 \rangle \xrightarrow{w_{t_1}^*} \langle t_1, n_2 \rangle \xrightarrow{\partial w_{t_2}} \langle t_1, t_2 \rangle = t.$$

Similarly, we have $w_t^* = w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A$, and the diagram is now:

$$\langle n, t' \rangle = \langle n_1, n_2, t' \rangle \xrightarrow{w_{t_1}^*} \langle t_1, n_2, t' \rangle \xrightarrow{\partial w_{t_2}^*} \langle t_1, t_2, t' \rangle \xrightarrow{A} \langle t_1 t_2, t' \rangle = \langle t, t' \rangle,$$

as is easily checked on Figure 7 bottom. □

So Condition (1.3) is satisfied. Then Proposition 1.4 tells us that a family of relations involving the generators A_α makes a presentation of $G(\mathcal{A})$ if and only if it contains enough relations to make the words $w_{t'}$ and $w_t \cdot A_\alpha$ equivalent whenever t' is the image of t under A_α. As we will show now, this is the case for the relations R_A of Proposition 2.8. Due to our inductive construction, it is convenient to prove two results simultaneously, namely one for w_t

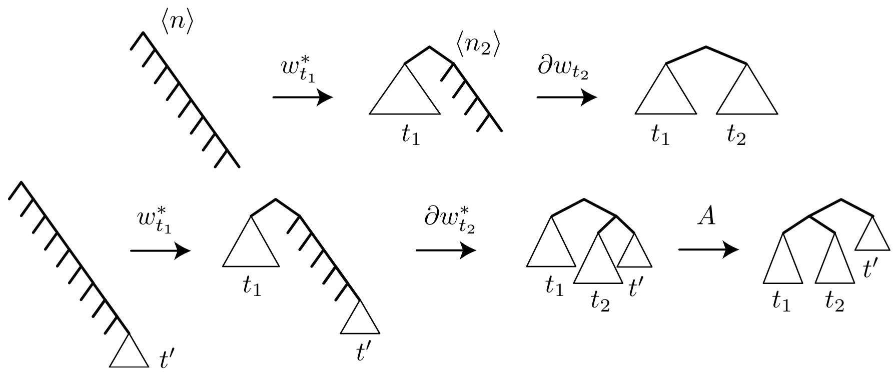


FIGURE 7. For t a tree of size n, the word w_t describes how to construct t from $\langle n\rangle$, and w^*_t describes how to construct $\langle t, t'\rangle$ from $\langle n, t'\rangle$; the figure illustrates the inductive argument for $t = t_1t_2$

and one for w^*_t. Note that the argument proving $\overline{w_{t'}} = \overline{w_t} \cdot A_\alpha$ when A_α maps t to t' similarly proves $\overline{w^*_{t'}} = \overline{w^*_t} \cdot A_{0\alpha}$, as, writing n for the common size of t and t', both operators map $\langle N\rangle$ to $\langle t', N-n\rangle$ for $N > n$. In the sequel, we use $\equiv_\square$ and $\equiv_{\pentagon}$ to indicate that we specifically use a geometric (*i.e.*, a twisted commutation) or a pentagon relation.

Lemma 2.12. *Assume $t' = t \bullet A_\alpha$. Then we have*

$$w_{t'} \equiv_{R_A} w_t \cdot A_\alpha \quad \text{and} \quad w^*_{t'} \equiv_{R_A} w^*_t \cdot A_{0\alpha}. \tag{2.9}$$

Proof. We use induction on the length of α as a sequence of 0's and 1's. Assume first that α is the empty address. The hypothesis that $t' = t \bullet A$ holds, *i.e.*, that the operator A maps t to t', means that there exist t_1, t_2, t_3 such that t is $t_1(t_2t_3)$ and t' is $(t_1t_2)t_3$. Then we find

$$w_{t'} = w^*_{t_1} \cdot \partial w^*_{t_2} \cdot A \cdot \partial w_{t_3} \equiv_\square w^*_{t_1} \cdot \partial w^*_{t_2} \cdot \partial^2 w_{t_3} \cdot A = w_t \cdot A,$$
$$w^*_{t'} = w^*_{t_1} \cdot \partial w^*_{t_2} \cdot A \cdot \partial w^*_{t_3} \cdot A \equiv_\square w^*_{t_1} \cdot \partial w^*_{t_2} \cdot \partial^2 w^*_{t_3} \cdot AA \equiv_{\pentagon} w^*_{t_1} \cdot \partial w^*_{t_2} \cdot \partial^2 w^*_{t_3} \cdot A_1AA_0 = w^*_t \cdot A_0.$$

Assume now $\alpha = 0\beta$. The hypothesis that A_α maps t to t' means that there exist t_1, t_2, t'_1 such that t is t_1t_2, t' is t'_1t_2, and A_β maps t_1 to t'_1. Using the induction hypothesis, we find

$$w_{t'} = w^*_{t'_1} \cdot \partial w_{t_2} \equiv_{(IH)} w^*_{t_1} \cdot A_{0\beta} \cdot \partial w_{t_2} \equiv_\square w^*_{t_1} \cdot \partial w_{t_2} \cdot A_{0\beta} = w_t \cdot A_\alpha,$$
$$w^*_{t'} = w^*_{t'_1} \cdot \partial w^*_{t_2} \cdot A \equiv_{(IH)} w^*_{t_1} \cdot A_{0\beta} \cdot \partial w^*_{t_2} \cdot A$$
$$\equiv_\square w^*_{t_1} \cdot \partial w^*_{t_2} \cdot A_{0\beta}A \equiv_\square w^*_{t_1} \cdot \partial w^*_{t_2} \cdot AA_{00\beta} = w^*_t \cdot A_{0\alpha}.$$

Finally, assume $\alpha = 1\beta$. With similar notation, we have $t = t_1t_2$ and $t' = t_1t'_2$ with A_β mapping t_2 to t'_2, and we find now

$$w_{t'} = w^*_{t_1} \cdot \partial w_{t'_2} \equiv_{(IH)} w^*_{t_1} \cdot \partial w_{t_2} \cdot A_{1\beta} = w_t \cdot A_\alpha,$$
$$w^*_{t'} = w^*_{t_1} \cdot \partial w^*_{t'_2} \cdot A \equiv_{(IH)} w^*_{t_1} \cdot \partial w^*_{t_2} \cdot A_{10\beta}A \equiv_\square w^*_{t_1} \cdot \partial w^*_{t_2} \cdot AA_{01\beta} = w^*_t \cdot A_{0\alpha},$$

which completes the proof. □

Applying Proposition 1.4, we deduce:

Proposition 2.13. *The relations R_A, i.e., the geometric relations for A plus the pentagon relations, make a presentation of the group $G(\mathcal{A})$, i.e., F, in terms of the generators A_α.*

2.5. **The standard presentation.** There is a well-known presentation of F in terms of an infinite sequence of generators, usually denoted x_i, indexed by nonnegative integers [5]. It is easy to establish the connection between these generators and our current generators A_α and, using Proposition 1.4 again, to re-obtain the standard presentation of F as a direct corollary.

Definition. (i) For $i \geqslant 1$, we put $a_i = A_{1^{i-1}}$, and we denote by $\boldsymbol{a}$ the family of all a_i's.

(ii) We denote by R_a the subfamily of R_A consisting of those relations in R_A that involve the generators of $\boldsymbol{a}$ exclusively, namely the relations $a_i a_{j-1} = a_j a_i$ for $j \geqslant i+2$.

Proposition 2.14. *The set $\boldsymbol{a}$ generates $G(\mathcal{A})$, i.e., F, and the relations R_a make a presentation of $G(\mathcal{A})$ in terms of the generators a_i.*

Proof. By construction, the words w_t belong to $W(\boldsymbol{a})$, and we can apply Proposition 1.4 to the family $\boldsymbol{a}$. So, in order to prove that R_a makes a presentation, it suffices to check that the relations of R_a are sufficient to establish the equivalence of $w_{t'}$ and $w_t \cdot a_i$ when a_i maps t to t'. Looking at the proof of Lemma 2.12 immediately shows that this is true. □

As the a_i's generates F, each A_α can be expressed in terms of the a_i's. For α an address containing at least one 0, say $\alpha = 1^p 00^{e_0} 10^{e_1} 1 \ldots 10^{e_q}$ with $p, q, e_0, \ldots, e_q \geqslant 0$, one can check

$$A_\alpha = (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \ldots a_{p+q+1}^{e_q+1})(a_{p+q+1} a_{p+q+2}^{-1})(a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \ldots a_{p+q+1}^{e_q+1})^{-1}.$$

For instance, for $\alpha = 01100$, we have $A_{01100} = a_1 a_2 a_3^4 a_4^{-1} a_3^{-3} a_2^{-1} a_1^{-1}$. As was noted in the introduction, the current a_i corresponds to x_{i-1}^{-1} in literature about F.

2.6. **The lattice structure of F.** It is known that F is a finitely presented group, generated by the two elements a_1, a_2. Using infinite presentations has disadvantages, and it may seem strange to replace the infinite family $\boldsymbol{a}$, which requires a very simple set of relations, with the still larger family $\boldsymbol{A}$ that involves a seemingly more complicated set of relations. However, one of the interests of the presentation $(\boldsymbol{A}; R_A)$ of F is that it is more symmetric, giving the same role to the left and right directions, contrary to $\boldsymbol{a}$ that priviledges the right one.

In particular, considering the generators A_α makes it natural to introduce the submonoid F^+ of F generated by these elements. Using a monoid version of Proposition 1.4 and a convenient combinatorial methods, one can show that F^+ admits, as a monoid, the presentation $(\boldsymbol{A}, R_A)$ and that it is isomorphic to the geometry monoid of oriented associativity $G^+(\mathcal{A})$ defined as $G(\mathcal{A})$ but considering the positive operators A_α only [6]. Contrary to the submonoid of F generated by the elements a_i, the monoid F^+ admits both left and right least common multiples, and one obtains in this way a double lattice structure on F.

Another interest of considering the generators A_α is that the associated Cayley graph is closely connected with Stasheff's associahedra: essentially, the graph is a direct limit of the associahedra, which appear as the orbits of the (partial) action of F on binary trees. These aspects will be investigated in a separate forthcoming paper [10].

3. The geometric presentation of Thompson's group V

Our approach to Thompson's group F was based on its connection with the associativity. We now develop a similar approach for Thompson's group V. The latter appears when the commutativity law $xy = yx$ is added. As in Section 2, the geometry of the commutativity operators leads to a natural presentation: in addition to the geometric and pentagon relations, the only new relations are the MacLane–Stasheff hexagon relations, plus some torsion relations.

3.1. **The geometry monoid of a family of algebraic laws.** The approach developed in Section 2 for the special case of associativity extends to arbitrary algebraic laws. The general form of an identity $\mathcal{I}$ is $\tau_- = \tau_+$, where τ_-, τ_+ are formal combinations of variables, or, equivalently, coloured trees. For each such $\mathcal{I}$, we can consider the partial operator I on $T_{\mathbf{N}}$ such that a tree t belongs to the domain of I if it can be written as τ_-^σ for some substitution σ, and, then, define $t \bullet I$ to be τ_+^σ. The operator I^{-1} is defined symmetrically, and, as above, we denote by $I_\alpha^{\pm 1}$ the translated copy $\partial_\alpha I^{\pm 1}$, *i.e.*, the result of letting $I^{\pm 1}$ act on the α-subtree.

Definition. For $\mathcal{I}, \mathcal{J}, \ldots$ algebraic laws, we define the *geometry monoid* of $\mathcal{I}, \mathcal{J}, \ldots$, denoted $\mathcal{G}(\mathcal{I}, \mathcal{J}, \ldots)$, to be the monoid generated by all partial operators $I_\alpha^{\pm 1}, J_\alpha^{\pm 1}, \ldots$

Thus, the monoid $\mathcal{G}(\mathcal{A})$ of Section 2 is the geometry monoid of the associativity law. Formally, the definition of the operators I_α and, therefore, of the geometry monoid, depends on the considered family of (coloured) trees. We shall forget about this here, which amounts to assuming that we work once for all inside a sufficiently large family of coloured trees $T_{\mathbf{N}}$.

In this framework, it is obvious that two trees t, t' are $\{\mathcal{I}, \mathcal{J}, \ldots\}$-equal if and only if some element of $\mathcal{G}(\mathcal{I}, \mathcal{J}, \ldots)$ maps t to t'. At this degree of generality, we cannot expect really deep results, and going further requires to restrict the considered laws. An unpleasant phenomenon is that, in general, the geometry monoid $\mathcal{G}(\mathcal{I}, \mathcal{J}, \ldots)$ contains the empty mapping, *i.e.*, there exist products of operators $I_\alpha^{\pm 1}$, $J_\beta^{\pm 1}$, ... applying to no tree, typically because the labels cannot be compatible. This however is excluded when the laws are simple enough.

Definition. A law $\tau_- = \tau_+$ is said to be *linear* if the same variables occur in τ_- or in τ_+ and each of them occurs exactly once.

The associativity law $x(yz) = (xy)z$ is linear, as x, y, and z occur only once on each side, while the self-distributivity law $x(yz) = (xy)(xz)$ is not, as x is repeated twice on the right.

Lemma 3.1. *Assume that $\mathcal{I}, \mathcal{J}, \ldots$ are linear laws. Then each operator in $\mathcal{G}(\mathcal{I}, \mathcal{J}, \ldots)$ admits a seed consisting of injective trees, i.e., there exists a pair of injective trees (t, t') such that, as a pair of trees, f is the set of all substitutes of (t, t').*

Proof. The point is that, if t_1, t_2 are injective trees, then there always exists substitutions σ_1, σ_2 such that $t_1^{\sigma_1}$ and $t_2^{\sigma_2}$ are equal, which need not be the case when some labels in t_1 or t_2 occur twice. Then the substitutions may be chosen so that the common skeleton of $t_1^{\sigma_1}$ and $t_2^{\sigma_2}$ is the union of the skeletons of t_1 and t_2, and the proof is the same as for Lemma 2.3. □

In the previous case, Lemma 2.2 applies, and, as in the case of $\mathcal{G}(\mathcal{A})$, it leads to a group.

Proposition 3.2. *Let $\mathcal{I}, \mathcal{J}, \ldots$ be linear algebraic laws. Then near-equality is a congruence on $\mathcal{G}(\mathcal{I}, \mathcal{J}, \ldots)$, and the quotient-monoid is a group. The operators $I_\alpha^{\pm 1}$, $J_\alpha^{\pm 1}$, ... induce a partial action of this group on trees. Injective trees form a discriminating family for this action.*

Definition. Under the above hypothesis, the group $\mathcal{G}(\mathcal{I}, \mathcal{J}, \ldots)/\!\approx$ is called the *geometry group* of the laws $\mathcal{I}, \mathcal{J}, \ldots$, and it is denoted $G(\mathcal{I}, \mathcal{J}, \ldots)$.

(Here, we restrict to laws that involve a single binary operation. A similar approach is of course possible for laws involving more than one operation, at the expense of considering trees in which the internal nodes are marked with operation symbols and their degree is adjusted to the arity of the operation.)

3.2. **Commutativity operators and Thompsons's group V.** The commutativity law

$$(\mathcal{C}) \qquad xy = yx$$

is eligible for the previous approach. Here the basic operator is the operator exchanging the left and the right subtrees of a tree:

Definition. We denote by C the (partial) operator that maps every tree of the form t_1t_2 to the corresponding tree t_2t_1. For each address α, we put $C_\alpha = \partial_\alpha C$. We define $\mathcal{G}(\mathcal{A}, \mathcal{C})$ to be the monoid generated by all operators A_α and C_α and their inverses.

Associativity and commutativity are linear laws, hence Lemma 3.1 and, therefore, Proposition 3.2 apply. So, near-equality is a congruence on the monoid $\mathcal{G}(\mathcal{A}, \mathcal{C})$, and we obtain a group, denoted $G(\mathcal{A}, \mathcal{C})$ by identifying near-equal operators. As in Section 2, we shall use A_α for the class of A_α in $G(\mathcal{A}, \mathcal{C})$, and, similarly, C_α for the class of C_α. We still denote by $\boldsymbol{A}$ the family of all A_α's, and, similarly, we use $\boldsymbol{C}$ for the family of all C_α's.

Proposition 3.3. *The group $G(\mathcal{A}, \mathcal{C})$ is (isomorphic to) Thompson's group V, i.e., V is the geometry group of associativity and commutativity.*

Proof. (Figure 8) We associate with each element of $G(\mathcal{A},\mathcal{C})$ an element of V, *i.e.*, a piecewise linear mapping of $[0,1]$ into itself as in we did for $G(\mathcal{A})$ and F in Section 2: we associate to each tree a dyadic partition of $[0,1]$, and we map f to the piecewise linear function that maps the partition associated to t' to the partition associated to t, where (t,t') is a seed for f—we again reverse the orientation to obtain a homomorphism with composition—and interpolates the values. The latter homomorphism is surjective since, as was shown in Section 2, its image includes F, and it contains the mappings denoted C and π_0 in [5], which correspond to AC_0A^{-1} and $AC_0A^{-1}C_1$ respectively. □

In the sequel, we identify $G(\mathcal{A},\mathcal{C})$ and V.

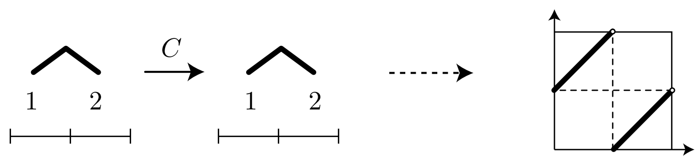


FIGURE 8. From $G(\mathcal{A},\mathcal{C})$ to V: the action of C

3.3. **Guessing relations in $\mathcal{G}(\mathcal{A},\mathcal{C})$.** As in the case of $\mathcal{G}(\mathcal{A})$, geometric inheritance provides twisted commutation relations in $\mathcal{G}(\mathcal{A},\mathcal{C})$, and therefore in the group $G(\mathcal{A},\mathcal{C})$. First, we observed that $\partial^2 f \cdot A = A \cdot \partial f$ holds for every mapping f, and used it for $f = A_\alpha$ to obtain $A_{11\alpha} \cdot A = A \cdot A_{1\alpha}$. Applying it now to $f = C_\alpha$, we deduce $C_{11\alpha} \cdot A = A \cdot C_{1\alpha}$ similarly. In this way, using X_α to represent either A_α or C_α, we obtain the following relations:

$$\begin{cases} X_\beta A_\alpha = A_\alpha X_\beta \quad \text{whenever } \beta \perp \alpha \text{ holds,} \\ X_{\alpha 11\beta} A_\alpha = A_\alpha X_{\alpha 1\beta}, \\ X_{\alpha 10\beta} A_\alpha = A_\alpha X_{\alpha 10\beta}, \\ X_{\alpha 0\beta} A_\alpha = A_\alpha X_{\alpha 00\beta}. \end{cases} \tag{3.1}$$

New inheritance phenomena appear with C_α: its action on a tree t exchanges the $\alpha 0$- and $\alpha 1$-subtrees of t, and we deduce the following relations, where X_α still stands for A_α or C_α:

$$\begin{cases} X_\beta C_\alpha = C_\alpha X_\beta \quad \text{whenever } \beta \perp \alpha \text{ holds,} \\ X_{\alpha 0\beta} C_\alpha = C_\alpha X_{\alpha 1\beta}, \\ X_{\alpha 1\beta} C_\alpha = C_\alpha X_{\alpha 0\beta}. \end{cases} \tag{3.2}$$

The relations mentioned in (3.1) and (3.2) will be called the A- and C-*geometric* relations, respectively. Apart from the geometric relations, we know that the pentagon relations, *i.e.*,

$$A_0 A A_1 = A^2 \tag{3.3}$$

and its shifted copies, are satisfied in $\mathcal{G}(\mathcal{A})$, hence in $\mathcal{G}(\mathcal{A},\mathcal{C})$. Two more types arise now.

Lemma 3.4. *The following relations and their translated copies hold in $\mathcal{G}(\mathcal{A},\mathcal{C})$:*

$$ACA = C_0 A C_1, \tag{3.4}$$

$$C^2 \approx \mathrm{id}. \tag{3.5}$$

The verification for (3.4), which corresponds to two ways of going from $(t_1t_2)t_3$ to $(t_2t_3)t_1$, is given in Figure 9. The involutivity of C is obvious—but, as C is defined only on those trees that are not $\bullet$, we obtain a $\approx$-relation, not an equality.

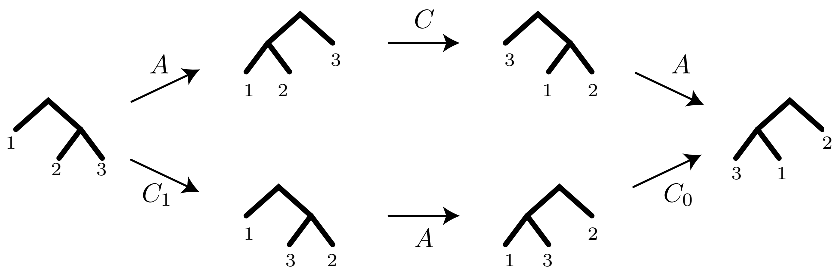


Figure 9. The hexagon relation

Definition. Let R_{AC} consist of all A- and C-geometric relations, *i.e.*, the translated copies of

$$(\square_\perp) \qquad X_{0\alpha} \cdot Y_{1\beta} = Y_{1\beta} \cdot X_{0\alpha},$$

$$(\square_A) \qquad X_{11\alpha} \cdot A = A \cdot X_{1\alpha}, \qquad X_{10\alpha} \cdot A = A \cdot X_{01\alpha}, \qquad X_{0\alpha} \cdot A = A \cdot X_{00\alpha},$$

$$(\square_C) \qquad X_{0\alpha} \cdot C = C \cdot X_{1\alpha}, \qquad X_{1\alpha} \cdot C = C \cdot X_{0\alpha},$$

with $X, Y = A$ or C, plus the pentagon relations, *i.e.*, the translated copies of

$$(\pentagon) \qquad AA = A_1 A A_0,$$

plus the hexagon relations, defined to be the translated copies of

$$(\hexagon) \qquad ACA = C_1 A C_0 \quad \text{and} \quad A^{-1} C A^{-1} = C_0 A^{-1} C_1.$$

The relations in $\mathcal{G}(\mathcal{A}, \mathcal{C})$ induce similar relations in $G(\mathcal{A}, \mathcal{C})$, *i.e.*, in V. So, we may state:

Proposition 3.5. *All relations in R_{AC} plus the torsion relations $C_\alpha^2 = 1$ are satisfied by the elements of $\boldsymbol{A}$ and $\boldsymbol{C}$ in the group $G(\mathcal{A}, \mathcal{C})$, i.e., in V.*

Distinguishing two hexagon relations, which are equivalent when the torsion relations $C_\alpha^2 = 1$ are present, may seem strange. The reason is that we consider a torsion-free version of V in Section 5, and it is convenient to keep track of the torsion relations from now on.

3.4. **Restricting the family of generators.** As in the case of F, we shall consider two families of generators for the group V: besides the families $\boldsymbol{A}$ and $\boldsymbol{C}$ comprising all A_α's and C_α's, we shall also consider the proper subfamilies corresponding to right branch addresses.

Definition. For $i \geqslant 1$, we put $c_i = C_{1^{i-1}}$. We denote by $\boldsymbol{c}$ the family of all c_i's.

Thus c_i is an exact counterpart to a_i. We now list some relations satisfied by the elements of $\boldsymbol{a}$ and $\boldsymbol{c}$ in $G(\mathcal{A}, \mathcal{C})$. A disadvantage of restricting the families of generators is that expressing the geometric phenomena is less simple than with the whole families $\boldsymbol{A}$ and $\boldsymbol{C}$.

Definition. We define R_{ac} to consist of the following relations:

$$a_i x_{j-1} = x_j a_i \qquad \text{for } j \geqslant i+2 \text{ and } x = a \text{ or } c, \tag{3.6}$$

$$c_i a_i^{-1} c_{i+1}^{-1} x_j = x_j c_i a_i^{-1} c_{i+1}^{-1} \qquad \text{for } j \geqslant i+2 \text{ and } x = a \text{ or } c, \tag{3.7}$$

$$a_{i+1} a_i c_i^e a_{i+1} = a_i^2 c_i^e \qquad \text{for } e = \pm 1, \tag{3.8}$$

$$a_i c_i c_{i+1} a_i = c_{i+1} c_i, \tag{3.9}$$

$$c_{i+1} c_i a_i^{-1} c_{i+1} = c_i a_i^{-1} c_i a_i^{-1}. \tag{3.10}$$

Lemma 3.6. *All relations in R_{ac} follow from R_{AC} (and the definitions $a_i = A_{1^{i-1}}$, $c_i = C_{1^{i-1}}$).*

Proof. It is sufficient to establish the relations for $i = 1$ and then use ∂^{i-1} to deduce the general version. Relations (3.6) and (3.7) are of purely geometric nature: (3.6) is a A-geometric relation, and (3.7) follows from

$$CA^{-1}C_1^{-1}X_{11\alpha} \equiv_\square CA^{-1}X_{10\alpha}C_1^{-1} \equiv_\square CX_{01\alpha}A^{-1}C_1^{-1} \equiv_\square X_{11\alpha}CA^{-1}C_1^{-1},$$

which is valid both for $X = A$ or C. Relations (3.8) use the pentagon relations:

$$A_1 A C^e A_1 \equiv_{\square} A_1 A A_0 C^e \equiv_{\pentagon} A^2 C^e.$$

Finally, appealing to the hexagon relations, we find

$$ACC_1 \equiv ACAA^{-1}C_1 \equiv_{\hexagon} C_1 AC_0 A^{-1} C_1 \equiv_{\hexagon} C_1 AA^{-1} CA^{-1} \equiv C_1 CA^{-1},$$
$$C_1 CA^{-1} C_1 \equiv_{\square} CC_0 A^{-1} C_1 \equiv_{\hexagon} CA^{-1} CA^{-1},$$

which gives (3.9) and (3.10). □

3.5. **Constructing trees.** We aim at proving that the relations R_{AC} and R_{ac} make presentations of the group V. As in the case of F, we shall use the criterion of Proposition 1.4. So, as in Section 2, the point is to introduce for each tree t a distinguished word w_t that describes the construction of t from some distinguished tree in its V-orbit.

In contrast to the case of associativity, considering commutativity requires that we take labels into account. Indeed, uncoloured trees are not discriminating: for instance, the operators id and C are not (near)-equal, but both fix $\bullet\bullet$. We use coloured versions of the right vines $\langle n \rangle$.

Definition. For $I_1, \ldots, I_k$ finite subsets of $\mathbf{N}$, we define the *coloured right vine* $\langle I_1,\ldots,I_k \rangle$ by

$$\langle I_1, \ldots, I_k \rangle = \bullet_{\ell_1}(\bullet_{\ell_2}(\ldots(\bullet_{\ell_{n-1}}\bullet_{\ell_n})\ldots)),$$

where $(\ell_1, \ldots, \ell_n)$ is the increasing enumeration of I_1, followed by the increasing enumeration of I_2, etc. (Figure 10).

FIGURE 10. The coloured right vines $\langle\{2, 5, 6\}, \{1, 3, 4\}\rangle$ and $\langle\{2, 5, 6, 1, 3, 4\}\rangle$; the latter is also $\langle\{1, 2, 3, 4, 5, 6\}\rangle$

In particular $\langle I \rangle$ is the vine in which the labels are the elements of I enumerated in increasing order. By construction, all coloured vines $\langle I \rangle$ are injective trees, so, clearly, we have:

Lemma 3.7. *Coloured vines make a discriminating family for the action of $G(\mathcal{A},\mathcal{C})$ on $T_{\mathbf{N}}$.*

The scheme is now the same as in Section 2: in order to apply Proposition 1.4, we select, for each tree t with labels I, a word w_t that describes how t can be constructed from the vine $\langle I \rangle$. For the skeleton, we can use associativity as in Section 2. For the labels, we use commutativity, *i.e.*, operators C_α. The first step for an inductive construction is to define an operator that maps $\langle I \cup J \rangle$ to $\langle I, J \rangle$ for disjoint I, J. To this end, we introduce new elements of $G(\mathcal{A},\mathcal{C})$.

Definition. (Figure 11) For each address α, we put $S_\alpha = C_\alpha A_\alpha^{-1} C_{\alpha 1}^{-1}$. We denote by $\boldsymbol{S}$ the family of all S_α's, and by R_{ACS} the family obtained by adding the definition of S_α to R_{AC}. For $i \geqslant 1$, we put $s_i = S_{1^{i-1}}$, *i.e.*, $s_i = c_i a_i^{-1} c_{i+1}^{-1}$, and we denote by $\boldsymbol{s}$ the family of all s_i's.

Definition. For I, J finite disjoint subsets of $\mathbf{N}$, the word $c_{I,J}$ is inductively determined by $c_{\varnothing,\varnothing} = \varepsilon$ and the rules: for ℓ smaller than all elements of I and J,

$$c_{\{\ell\}\cup I,J} = \partial c_{I,J}, \quad c_{I,\{\ell\}\cup J} = \begin{cases} s_1 s_2 \ldots s_{p-1} c_p & \text{if } I \text{ has } p \text{ elements and } J \text{ is empty,} \\ \partial c_{I,J} \cdot s_1 s_2 \ldots s_p & \text{if } I \text{ has } p \text{ elements and } J \text{ is nonempty.} \end{cases}$$

The word $s_{I,J}$ is defined similarly, except that $c_{I,\{\ell\}}$ is defined to be $s_1 s_2 \ldots s_p$.

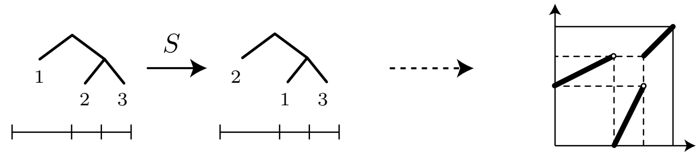


FIGURE 11. The action of S; the operator s_i, *i.e.*, $S_{1^{i-1}}$, switches the ith and the $(i+1)$th factors of $\langle t_1, \ldots, t_n \rangle$, as it maps the latter to $\langle t_1, \ldots, t_{i+1}, t_i, \ldots t_n \rangle$.

Example 3.8. Let $I = \{2,5,6\}$ and $J = \{1,3,4\}$. By considering the elements of $I \cup J$ in decreasing order, we find successively $c_{\varnothing,\varnothing} = \varepsilon$, $c_{\{6\},\varnothing} = \varepsilon$, $c_{\{5,6\},\varnothing} = \varepsilon$, $c_{\{5,6\},\{4\}} = s_1 c_2$, $c_{\{5,6\},\{3,4\}} = \partial(s_1 c_2) \cdot s_1 s_2 = s_2 c_3 s_1 s_2$, $c_{\{2,5,6\},\{3,4\}} = \partial(s_2 c_3 s_1 s_2) = s_3 c_4 s_2 s_3$, $c_{\{2,5,6\},\{1,3,4\}} = \partial(s_3 c_4 s_2 s_3) \cdot s_1 s_2 s_3 = s_4 c_5 s_3 s_4 s_1 s_2 s_3$, and $s_{\{2,5,6\},\{1,3,4\}} = s_4 s_5 s_3 s_4 s_1 s_2 s_3$.

Lemma 3.9. *For all sets I, J, and for every tree t, we have*

$$\langle I \cup J \rangle \xrightarrow{c_{I,J}} \langle I, J \rangle \qquad \textit{and} \qquad \langle I \cup J, t \rangle \xrightarrow{s_{I,J}} \langle I, J, t \rangle.$$

Proof. We use induction on the cardinality of $I \cup J$. The result is clear if I and J are empty. Assume that ℓ is smaller than all elements in I and J. The induction hypothesis asserts that $c_{I,J}$ maps $\langle I \cup J \rangle$ to $\langle I, J \rangle$, hence $\partial c_{I,J}$ maps $\bullet_\ell \langle I \cup J \rangle$, which is $\langle \{\ell\} \cup I \cup J \rangle$, to $\bullet_\ell \langle I, J \rangle$, *i.e.*, to $\langle \{\ell\} \cup I, J \rangle$, as expected for $c_{\{\ell\} \cup I, J}$.

Let us consider $c_{I,\{\ell\} \cup J}$. Let p be the cardinal of I. Assume first $J \neq \emptyset$. We have seen that $\partial c_{I,J}$ maps $\langle \{\ell\} \cup I \cup J \rangle$ to $\langle \{\ell\} \cup I, J \rangle$. Then the iterated transposition $s_1 s_2 \ldots s_p$ carries the leftmost leaf of $\langle \{\ell\} \cup I, J \rangle$, *i.e.*, $\bullet_\ell$, through p leaves to the right, *i.e.*, we obtain $\langle I, \{\ell\}, J \rangle$, which is also $\langle I, \{\ell\} \cup J \rangle$. Finally, if J is empty, then $c_{I,J}$ is ε, as an induction shows, and $s_1 s_2 \ldots s_{p-1} c_p$ maps $\langle \{\ell\}, I \rangle$ to $\langle I, \{\ell\} \rangle$. So, in each case, $c_{I,\{\ell\} \cup J}$ maps $\langle \{\ell\} \cup I \cup J \rangle$ to $\langle I, \{\ell\} \cup J \rangle$.

The argument is similar for $s_{I,J}$. □

We are now ready for defining a word w_t that describes how to construct a coloured tree t with labels I from the right vine $\langle I \rangle$. The current construction is similar to that of Section 2. The only change is that, in the induction step, we first sort the labels in order to push to the initial positions the labels that correspond to the left subtree. This is exactly what (the operators associated with) $c_{I,J}$ and $s_{I,J}$ do. So the following definition should be natural.

Definition. For each injective tree t, the words w_t, w_t^* are defined by the rules:

$$w_t = w_t^* = \varepsilon \qquad \text{for } t \text{ of size } 1,$$
$$w_t = c_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}, \quad w_t^* = s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \qquad \text{for } t = t_1 t_2 \text{ and } I_k \text{ the labels in } t_k.$$

The following result is the exact counterpart to Lemma 2.11. For $I = \{\ell_1, \ldots, \ell_n\}$ and t a tree, we use $\langle I, t \rangle$ for $\langle \bullet_{\ell_1}, \ldots, \bullet_{\ell_n}, t \rangle$.

Lemma 3.10. *For each injective tree t with labels I, and each tree t', we have*

$$\langle I \rangle \xrightarrow{w_t} t \qquad \textit{and} \qquad \langle I, t' \rangle \xrightarrow{w_t^*} \langle t, t' \rangle, \tag{3.11}$$

i.e., w_t constructs t from $\langle I \rangle$, and w_t^* constructs $\langle t, t' \rangle$ from $\langle I, t' \rangle$.

Proof. The inductive verification is the same as for Lemma 2.11. The diagrams are now:

$$\langle I \rangle = \langle I_1 \cup I_2 \rangle \xrightarrow{c_{I_1,I_2}} \langle I_1, I_2 \rangle \xrightarrow{w_{t_1}^*} \langle t_1, I_2 \rangle \xrightarrow{\partial w_{t_2}} \langle t_1, t_2 \rangle = t,$$
$$\langle I, t' \rangle = \langle I_1 \cup I_2, t' \rangle \xrightarrow{s_{I_1,I_2}} \langle I_1, I_2, t' \rangle \xrightarrow{w_{t_1}^*} \langle t_1, I_2, t' \rangle$$
$$\xrightarrow{\partial w_{t_2}^*} \langle t_1, t_2, t' \rangle \xrightarrow{A} \langle t_1 t_2, t' \rangle = \langle t, t' \rangle$$

for $t = t_1 t_2$ and I_1, I_2 the sets of labels in t_1 and t_2 respectively. □

3.6. **Derived relations.** In order to apply Proposition 1.4 and prove that the relations of Proposition 3.5 make a presentation of the group $G(\mathcal{A},\mathcal{C})$, *i.e.*, of V, we have to check that there are enough relations to establish the equivalence of $w_{t'}$ and $w_t \cdot X_\alpha$ whenever X_α maps t to t', where X is either A or C. The needed verifications are easy, but longer than in the case of $G(\mathcal{A})$, and we begin with some technical, but easy preparatory results asserting that certain relations involving the letters A_α, C_α, and S_α follow from R_{ACS}.

Lemma 3.11. *The following relations follow from R_{ACS}:*

(i) The A- and C-geometric relations of R_{AC} in which X or Y is replaced with S;

(ii) The S-geometric relations, defined to be the translated copies of

$$(\Box_S) \qquad X_{11\alpha} \cdot S = S \cdot X_{11\alpha}, \qquad X_{10\alpha} \cdot S = S \cdot X_{0\alpha}, \qquad X_{0\alpha} \cdot S = S \cdot X_{10\alpha},$$

in which X stands for A, C or S,

(iii) The translated copies of the relations

$$SA = AC_0, \qquad SA_1A = A_1AS_0, \tag{3.12}$$

$$S_1SA_1 = AS, \quad SS_1A = A_1S, \quad SS_1S = S_1SS_1. \tag{3.13}$$

Proof. The extension of the A- and C-geometric relations to S_α is obvious, as S_α is defined from C_α, A_α, and $C_{1\alpha}$. The S-geometric relations follow from the other geometric relations. For instance, we find

$$SX_{11\alpha} = AC_0A^{-1}X_{11\alpha} \equiv_{\Box_A} AC_0X_{1\alpha}A^{-1} \equiv_{\Box_\perp} AX_{1\alpha}C_0A^{-1} \equiv_{\Box_A} X_{11\alpha}AC_0A^{-1} = X_{11\alpha}S.$$

The first relation in (3.12) follows from the definition and an hexagon relation:

$$SA_0 = CA^{-1}C_1^{-1}A \equiv_{\hexagon} AC_0.$$

The second relation comes by cancelling A_0 on the right in

$$SA_1AA_0 \equiv_{\pentagon} SAA \equiv_{(3.12)} AC_0A \equiv_{\Box} AAC_{00} \equiv_{\pentagon} A_1AA_0C_{00} \equiv_{(3.12)} A_1AS_0A_0.$$

Then we observe that the hexagon relation implies

$$C_1S \equiv C_1SAA^{-1} \equiv_{(3.12)} C_1AC_0A^{-1} \equiv_{\hexagon} ACAA^{-1} \equiv AC. \tag{3.14}$$

Next, the first two relations in (3.13) are obtained by cancelling A on the right in

$$\begin{aligned} S_1SA_1A &\equiv_{(3.12)} S_1A_1AS_0 \equiv_{(3.12)} A_1C_{10}AS_0 \\ &\equiv_{\Box} A_1AC_{01}S_0 \equiv_{(3.14)} A_1AA_0C_0 \equiv_{\pentagon} AAC_0 \equiv_{(3.12)} ASA, \\ SS_1AA &\equiv_{\hexagon} SS_1A_1AA_0 \equiv_{(3.12)} SA_1C_{10}AA_0 \equiv_{\Box} SA_1AC_{01}A_0 \\ &\equiv_{(3.12)} A_1AS_0C_{01}A_0 \equiv A_1AC_0 \equiv_{(3.12)} A_1SA. \end{aligned}$$

Finally, we have

$$SC_1S \equiv_{(3.14)} SAC \equiv_{(3.12)} AC_0C \equiv_{\Box} ACC_1 \equiv_{(3.14)} C_1SC_1,$$

so, using $SA_1A \equiv_{(3.12)} A_1AS_0$, and $S_1A_1A \equiv_{(3.12)} A_1C_{10}A \equiv_{\Box} A_1AC_{01}$, we deduce

$$A_1ASS_1S \equiv \partial_0(SC_1S) \cdot A_1A \equiv \partial_0(C_1SC_1) \cdot A_1A \equiv A_1AS_1SS_1,$$

which implies the third relation in (3.13) by cancelling A_1A on the left. ☐

On the other hand, we observe that, by construction, the words w_t and w_t^* involve the letters a_i, c_i, and s_i only. So it will be convenient to work with the following restricted list.

Definition. We define R_{acs} to consist of the following relations:

$$a_ix_{j-1} = x_ja_i, \qquad \text{with } j \geqslant i+2 \text{ and } x = a, c \text{ or } s, \tag{3.15}$$

$$s_ix_j = x_js_i, \qquad \text{with } j \geqslant i+2 \text{ and } x = a, c \text{ or } s, \tag{3.16}$$

$$s_is_{i+1}a_i = a_{i+1}s_i, \qquad \text{and} \qquad s_{i+1}s_ia_{i+1} = a_is_i, \tag{3.17}$$

$$s_ix_{i+1}s_i = x_{i+1}s_ix_{i+1}, \qquad \text{with } x = s \text{ or } c. \tag{3.18}$$

Lemma 3.12. *All relations in R_{acs} are consequences of R_{ac}, hence of R_{AC} (plus the definitions of a_i, c_i and s_i). Furthermore, $s_i^2 = 1$ follows from R_{ac} completed with the relations $c_i^2 = 1$.*

Proof. When x is a or c, (3.15) coincides with (3.6); for $x_j = s_j$, we apply (3.6) to c_j, a_j^{-1}, and c_{j+1}^{-1} successively. Similarly, (3.16) for $x_j = a_j$ or c_j directly follows from (3.7) owing to the definition of s_i; the relation for $x_j = s_j$ then follows by replacing s_j with its definition. As for (3.17), we find

$$s_i s_{i+1} a_i \equiv c_i a_i^{-1} a_{i+1}^{-1} c_{i+2}^{-1} a_i \equiv_{(3.6)} c_i a_i^{-1} a_{i+1}^{-1} a_i c_{i+1}^{-1} \equiv_{(3.8)} a_{i+1} c_i a_i^{-1} c_{i+1}^{-1} \equiv_{(3.19)} a_{i+1} s_i.$$

Next, we observe that the relation (3.9) of R_{ac} implies

$$s_i = c_i a_i^{-1} c_{i+1}^{-1} \equiv_{R_{ac}} c_{i+1}^{-1} a_i c_i, \tag{3.19}$$

and we deduce symmetrically

$$s_{i+1} s_i a_{i+1} \equiv c_{i+2}^{-1} a_{i+1} a_i c_i a_{i+1} \equiv_{(3.8)} c_{i+2}^{-1} a_i^2 c_i a_{i+1} \equiv_{(3.6)} a_i c_{i+1}^{-1} a_i c_i a_{i+1} = a_i s_i.$$

For (3.18) with $x_2 = c_2$, we have

$$s_1 c_2 s_1 \equiv c_1 a_1^{-1} s_1 = c_1 a_1^{-1} c_1 a_1^{-1} c_2^{-1} \equiv_{(3.10)} c_2 c_1 a_1^{-1} \equiv c_2 s_1 c_2.$$

As for (3.18) with $x_2 = s_2$, we have

$$\begin{aligned} s_1 s_2 s_1 &= s_1 c_2 a_2^{-1} c_3^{-1} s_1 \equiv_{(3.16)} s_1 c_2 a_2^{-1} s_1 c_3^{-1} \equiv_{(3.17)} s_1 c_2 s_1 a_1^{-1} s_2^{-1} c_3^{-1}, \\ s_2 s_1 s_2 &= c_2 a_2^{-1} c_3^{-1} s_1 s_2 \equiv_{(3.16)} c_2 a_2^{-1} s_1 c_3^{-1} s_2 \equiv_{(3.17)} c_2 s_1 a_1^{-1} s_2^{-1} c_3^{-1} s_2 \\ &\equiv c_2 s_1 c_2 c_2^{-1} a_1^{-1} s_2^{-1} c_3^{-1} s_2 \equiv_{(3.15)} c_2 s_1 c_2 a_1^{-1} c_3^{-1} s_2^{-1} c_3^{-1} s_2. \end{aligned}$$

Applying the relations $s_1 c_2 s_1 \equiv c_2 s_1 c_2$ and $s_2 c_3 s_2 \equiv c_3 s_2 c_3$—hence $s_2^{-1} c_3^{-1} \equiv c_3^{-1} s_2^{-1} c_3^{-1} s_2$—which were established above, we deduce $s_1 s_2 s_1 \equiv s_2 s_1 s_2$.

Finally, we have seen that R_{ac} implies $s_1 = c_1 a_1^{-1} c_2^{-1} \equiv c_2^{-1} a_1 c_1$, hence $s_1^2 \equiv c_1 c_2^{-2} c_1$: so $c_1^2 \equiv c_2^2 \equiv 1$ implies $s_1^2 \equiv 1$. ☐

For furure inductive arguments, we need some results about the auxiliary words $c_{I,J}$ and $s_{I,J}$.

Lemma 3.13. *For I, J, K disjoint with $p = \#I \geqslant 1$, we have*

$$x_{I \cup J, K} \cdot s_{I,J} \equiv_{R_{acs}} x_{I, J \cup K} \cdot \partial^p x_{J,K} \qquad \text{for } x = s \text{ and } x = c. \tag{3.20}$$

Proof. We begin with the auxiliary formulas

$$s_1 s_2 \dots c_{k+1} s_i \equiv_{R_{acs}} c_{i+1} s_1 s_2 \dots c_{k+1}, \quad \text{for } 1 \leqslant i \leqslant k, \tag{3.21}$$

$$s_1 s_2 \dots s_k c_{k+1} s_k \equiv_{R_{acs}} c_{k+1} s_1 s_2 \dots s_k c_{k+1}, \quad \text{for } 1 \leqslant k. \tag{3.22}$$

A direct inductive verification is possible; we can also observe that Lemma 3.12 shows that $(s_1, \dots, s_{k+1})$ and $(s_1, \dots, s_k, c_{k+1})$ satisfy the relations of Artin's presentation of the braid group B_{k+2}: therefore, every braid relation between the standard generators σ_i of B_{k+2} must hold between the s_i's, which is the case for the counterpart of (3.21) and (3.22).

Next, we claim that the following relations are true, where q denotes $\#J$:

$$s_1 s_2 \dots s_{q+r} s_{J,K} \equiv_{R_{acs}} \partial s_{J,K} \cdot s_1 s_2 \dots s_{q+r}, \tag{3.23}$$

$$s_1 s_2 \dots s_{q+r-1} c_{q+r} s_{J,K} \equiv_{R_{acs}} \partial c_{J,K} s_1 s_2 \dots s_{q+r-1} c_{q+r}. \tag{3.24}$$

Indeed, an easy induction shows that the word $c_{J,K}$ is a product of s_i's with $1 \leqslant i \leqslant q + r - 2$, and, if it not empty, of c_{q+r-1} occurring once, and that $s_{J,K}$ is obtained from $c_{I,J}$ by replacing the possible c_{q+r-1} with s_{q+r-1}. Then (3.23) comes by applying (3.21) with $k = q + r - 1$ to the letters s_i in $c_{J,K}$, and so does (3.24) using (3.22) for the possible letter c_{q+r-1} of $s_{J,K}$.

We turn to the first formula in (3.20). The result is trivial for $I = J = K = \emptyset$. For an induction, it is sufficient to prove that, if ℓ is smaller than all elements in $I \cup J \cup K$, the result is true for $(\{\ell\}, J, K)$, and it is true for $(\{\ell\} \cup I, J, K)$, $(I, \{\ell\} \cup J, K)$, and $(I, J, \{\ell\} \cup K)$ whenever it is for (I, J, K) and I is nonempty. For the case of $(\{\ell\}, J, K)$, we find

$$\begin{aligned} c_{\{\ell\}, J\cup K} \cdot s_{J,K} &= s_1 \ldots s_{q+r-1} c_{q+r} \cdot s_{J,K} \\ &\equiv_{(3.24)} \partial c_{J,K} \cdot s_1 \ldots s_{q+r-1} c_{q+r} \\ &= \partial c_{J,K} \cdot s_1 \ldots s_r \cdot \partial^r (s_1 \ldots s_{q-1} c_q) = c_{\{\ell\}\cup J,K} \cdot \partial^r c_{\{\ell\},J}. \end{aligned}$$

Assume $I \neq \emptyset$. For $(\{\ell\} \cup I, J, K)$, using the induction hypothesis, we find

$$\begin{aligned} c_{I\cup\{\ell\}, J\cup K} s_{J,K} &= \partial c_{I, J\cup K} \cdot s_1 s_2 \ldots s_{q+r} \cdot s_{J,K} \equiv_{(3.23)} \partial c_{I,J\cup K} \cdot \partial s_{J,K} \cdot s_1 s_2 \ldots s_{q+r} \\ &\equiv_{(IH)} \partial s_{I\cup J,K} \cdot \partial^{r+1} c_{I,J} \cdot s_1 s_2 \ldots s_{q+r} \\ &\equiv_{(3.16)} \partial s_{I\cup J,K} \cdot s_1 s_2 \ldots s_r \cdot \partial^{r+1} c_{I,J} \cdot s_{r+1} \ldots s_{q+r} \\ &= \partial s_{I\cup J,K} \cdot s_1 s_2 \ldots s_r \cdot \partial^r (\partial c_{I,J} \cdot s_1 s_2 \ldots s_q) = c_{\{\ell\}\cup I\cup J,K} \cdot \partial^r c_{I\cup\{\ell\},J}, \end{aligned}$$

The remaining cases are easy:

$$\begin{aligned} c_{I,\{\ell\}\cup J\cup K} \cdot s_{\{\ell\}\cup J,K} &= \partial c_{I,J\cup K} \cdot \partial s_{J,K} \cdot s_1 s_2 \ldots s_r \equiv_{(IH)} \partial s_{I\cup J,K} \cdot \partial^{r+1} c_{I,J} \cdot s_1 s_2 \ldots s_r \\ &\equiv_{(3.16)} \partial s_{I\cup J,K} \cdot s_1 s_2 \ldots s_r \cdot \partial^{r+1} c_{I,J} = c_{I\cup\{\ell\}\cup J,K} \cdot \partial^r c_{I,\{\ell\}\cup J}; \\ c_{I,J\cup\{\ell\}\cup K} s_{J,\{\ell\}\cup K} &= \partial c_{I,J\cup K} \cdot \partial s_{J,K} \equiv_{(IH)} \partial c_{I\cup J,K} \cdot \partial^r c_{I,J} = c_{I\cup J,\{\ell\}\cup K} \cdot \partial^{r+1} c_{I,J}, \end{aligned}$$

and the proof is complete. □

Definition. For $p, q \geqslant 1$, we put $c_{p,q} = c_{\{q+1,\ldots,q+p\},\{1,\ldots,q\}}$ and $s_{p,q} = s_{\{q+1,\ldots,q+p\},\{1,\ldots,q\}}$.

So $s_{p,q}$ is the iterated transposition that switches two blocks of p and q elements respectively, putting the p elements on the top. For instance, we have $s_{p,1} = s_1 \ldots s_{p-1}$, and $s_{1,q} = s_{q-1} \ldots s_1$.

Lemma 3.14. *For all p, q, r, we have*

$$c_{p+q,r} \equiv_{R_{acs}} s_{p,r} \cdot \partial^p c_{q,r} \quad \text{and} \quad s_{p+q,r} \equiv_{R_{acs}} s_{p,r} \cdot \partial^p s_{q,r}, \tag{3.25}$$

$$c_{p,q+r} \equiv_{R_{acs}} \partial^q c_{p,r} \cdot s_{p,q} \quad \text{and} \quad s_{p,q+r} \equiv_{R_{acs}} \partial^q s_{p,r} \cdot s_{p,q}, \tag{3.26}$$

$$a_{q+1} \cdot s_{p+1,q} \equiv_{R_{acs}} s_{p+2,q} \cdot a_1. \tag{3.27}$$

Proof. Relation (3.25) and (3.26) follow from (3.20) by taking $I = \{r+1, \ldots, r+p\}$, $J = \{r+p+1, \ldots, r+p+q\}$, $K = \{1, \ldots, r\}$, and $I = \{q+r+1, \ldots, q+r+p\}$, $J = \{1, \ldots, q\}$, $K = \{q+1, \ldots, q+r\}$, respectively. In the first case, we have $c_{I,J\cup K} = s_{p,r}$, and, in the second one, we have $c_{I\cup J,K} = \partial^q c_{p,r}$. For (3.27), we use induction. For $q = 0$, the result is clear; for $q \geqslant 1$, we find

$$\begin{aligned} a_{q+1} s_{p+1,q} &\equiv_{(3.25)} a_{q+1} \cdot \partial s_{p+1,q-1} \cdot s_{p+1,1} = \partial(a_q s_{p-1,q-1}) \cdot s_{p+1,1} \\ &\equiv_{(IH)} \partial s_{p+2,q-1} a_1 \cdot s_{p+1,1} = \partial s_{p+2,q-1} \cdot a_2 s_1 \ldots s_{p+1} \\ &\equiv_{(3.17)} \partial s_{p+2,q-1} \cdot s_1 s_2 a_1 s_2 \ldots s_{p+1} \\ &\equiv_{R_{acs}} \partial s_{p+2,q-1} \cdot s_1 s_2 s_3 \ldots s_{p+2} a_1 \equiv_{(3.25)} s_{p+2,q} a_1, \end{aligned}$$

which completes the computation. □

Lemma 3.15. *Assume that t is a size n tree. Then, for $p, q \geqslant 0$, we have*

$$x_{i+n} \cdot w_t^* \equiv_{R_{acs}} w_t^* \cdot x_{i+1}, \qquad \text{for } x = a, c \text{ or } s, \tag{3.28}$$

$$\partial^q w_t \cdot c_{1,q} \equiv_{R_{acs}} c_{n,q} \cdot w_t^* \quad \text{and} \quad \partial^q w_t^* \cdot s_{p+1,q} \equiv_{R_{acs}} s_{p+n,q} \cdot w_t^*, \tag{3.29}$$

$$w_t^* \cdot c_{p,q+1} \equiv_{R_{acs}} c_{p,q+n} \cdot \partial^q w_t, \quad \text{and} \quad w_t^* \cdot s_{p,q+1} \equiv_{R_{acs}} s_{p,q+n} \cdot \partial^q w_t^*. \tag{3.30}$$

Proof. We use induction on n. For $n = 1$, the words w_t and w_t^* are empty, and all relations are equalities. Otherwise, assume $t = t_1 t_2$, with, as usual, n_k the size of t_k and I_k its set of labels. For (3.28), we find

$$\begin{aligned} x_{i+n} \cdot w_t^* &= x_{i+n} \cdot s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \equiv_\Box s_{I_1,I_2} \cdot x_{i+n} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \\ &\equiv_{(IH)} s_{I_1,I_2} \cdot w_{t_1}^* \cdot x_{i+n_2} \cdot \partial w_{t_2}^* \cdot A \\ &\equiv_{(IH)} s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot x_{i+2} \cdot A \equiv_\Box s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \cdot x_{i+1} = w_t^* \cdot x_{i+1}. \end{aligned}$$

(The first equivalence holds because we consider s_{I_1,I_2}, which consists of s_i's only.)

We turn to the second relation in (3.29). Then the expected relation follows from the commutativity of the following diagram

$$\begin{array}{ccccccccc} \langle q,I,p,t'\rangle & \xrightarrow{\partial^q s_{I_1,I_2}} & \langle q,I_1,I_2,p,t'\rangle & \xrightarrow{\partial^q w_{t_1}^*} & \langle q,t_1,I_2,p,t'\rangle & \xrightarrow{\partial^{q+1} w_{t_2}^*} & \langle q,t_1,I_2,p,t'\rangle & \xrightarrow{a_{q+1}} & \langle q,t_1t_2,p,t'\rangle \\ \downarrow{\scriptstyle s_{p+n,q}} & & \downarrow{\scriptstyle s_{p+n,q}} & & \downarrow{\scriptstyle s_{p+n_2+1,q}} & & \downarrow{\scriptstyle s_{p+2,q}} & & \downarrow{\scriptstyle s_{p+1,q}} \\ \langle I,p,q,t'\rangle & \xrightarrow{s_{I_1,I_2}} & \langle I_1,I_2,p,q,t'\rangle & \xrightarrow{w_{t_1}^*} & \langle t_1,I_2,p,q,t'\rangle & \xrightarrow{\partial w_{t_2}^*} & \langle t_1,I_2,p,q,t'\rangle & \xrightarrow{a_1} & \langle t_1t_2,p,q,t'\rangle \end{array}$$

The first (leftmost) square is commutative by (3.25). The second one is commutative by induction hypothesis. For the third, (3.25) tells us that $s_{p+n_2+1,q}$ is R_{acs}-equivalent to $s_{1,q} \cdot \partial s_{p+n_2,q}$, and that $s_{p+2,q}$ is R_{acs}-equivalent to $s_{1,q} \cdot \partial s_{p+1,q}$. As $\partial^{q+1} w_{t_2}^*$ R_{acs}-commutes with $s_{1,q}$ by geometric relations, we are left with proving the R_{acs}-equivalence of $\partial^q w_{t_2}^* \cdot s_{p+1,q}$ and $s_{pn_2,q} \cdot \partial w_{t_2}^*$, which is the induction hypothesis. Finally, the commutativity of the last square follows from (3.27).

The verification of the other three formulas is similar. $\Box$

We are now in position for proving the counterpart to Lemma 2.12:

Lemma 3.16. *Assume $t' = t \bullet X_\alpha$, where X is A, C, or S. Then we have*

$$w_{t'} \equiv_{R_{ACS}} w_t \cdot X_\alpha \quad \text{and} \quad w_{t'}^* \equiv_{R_{ACS}} w_t^* \cdot X_{0\alpha}. \tag{3.31}$$

Proof. Clearly, it suffices to consider the cases of A_α and C_α, as S_α is defined from the latter. As for Lemma 2.12, we use induction on the length of α as a sequence of 0's and 1's. So assume first that α is the empty address. Let us consider the case of A. The hypothesis $t' = t \bullet A$ implies that there exist trees t_1, t_2, t_3 such that t is $(t_1t_2)t_3$, and t' is $t_1(t_2t_3)$. We write I_1 (*resp.* I_2, I_3) for the labels in t_1 (*resp.* t_2, t_3), and n_1 (*resp.* n_2, n_3) for their size. We obtain

$$w_{t'} = c_{I_1 \cup I_2, I_3} \cdot s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \cdot \partial w_{t_3} \tag{3.32}$$

$$w_t \cdot A = s_{I_1, I_2 \cup I_3} \cdot w_{t_1}^* \cdot \partial c_{I_2,I_3} \cdot \partial w_{t_2}^* \cdot \partial^2 w_{t_3} \cdot A \tag{3.33}$$

$$w_{t'}^* = s_{I_1 \cup I_2, I_3} \cdot s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot A \cdot \partial w_{t_3}^* \cdot A \tag{3.34}$$

$$w_t^* \cdot A_0 = s_{I_1, I_2 \cup I_3} \cdot w_{t_1}^* \cdot \partial s_{I_2,I_3} \cdot \partial w_{t_2}^* \cdot \partial^2 w_{t_3}^* \cdot A_1 A A_0. \tag{3.35}$$

Using geometric relations, we may move the factor A to the right in (3.32), while, in (3.33), we may replace $w_{t_1}^* \cdot \partial c_{I_2,I_3}$ with $\partial^p c_{I_2,I_3} \cdot w_{t_1}^*$ using (3.28). Then, applying (3.25) gives the equivalence of $w_{t'}$ and $w_t \cdot A$. The argument is similar for (3.34) and (3.35), the only difference being an additional pentagon relation for replacing AA by A_1AA_0 on the right.

For C, with similar notation, we have $t = t_1t_2$ and $t' = t_2t_1$, and we find now

$$w_{t'} = c_{I_2,I_1} \cdot w_{t_2}^* \cdot \partial w_{t_1}, \tag{3.36}$$

$$w_t \cdot C = c_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2} \cdot C, \tag{3.37}$$

$$w_{t'}^* = s_{I_2,I_1} \cdot w_{t_2}^* \cdot \partial w_{t_1}^* \cdot A, \tag{3.38}$$

$$w_t^* \cdot C_0 = s_{I_1,I_2} \cdot w_{t_1}^* \cdot \partial w_{t_2}^* \cdot AC_0. \tag{3.39}$$

By (3.28), we have $w^*_{t_2} \cdot \partial w_{t_1} \equiv_{R_{acs}} \partial^{n_2} w_{t_1} \cdot w^*_{t_2}$, and the R_{acs}-equivalence of $w_{t'}$ and $w_t \cdot C$ follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
\langle I\rangle & \xrightarrow{c_{I_1,I_2}} & \langle I_1,I_2\rangle & \xrightarrow{w^*_{t_1}} & \langle t_1,I_2\rangle & \xrightarrow{\partial w_{t_2}} & t_1t_2\\
\Vert & R_{acs}+\text{torsion} & \downarrow c_{n_2,n_1} & (3.30) & \downarrow c_{n_2,1} & (3.29) & \downarrow c_{1,1}=C\\
\langle I\rangle & \xrightarrow{c_{I_2,I_1}} & \langle I_2,I_1\rangle & \xrightarrow{\partial^{n_2}w_{t_1}} & \langle I_2,t_1\rangle & \xrightarrow{w^*_{t_2}} & t_2t_1
\end{array}$$

The commutativity of the left square follow from the fact that both $c_{I_1,I_2} \cdot c_{n_2,n_1}$ and c_{I_2,I_1} induce the same permutation of the labels: it follows that these words must be equivalent with respect to any family of relations that makes a presentation of the symmetric group, and, therefore, they are equivalent under the Coxeter relations of R_{acs} completed with the torsion relations $c_i^2 = s_i^2 = 1$.

The argument is similar for $w^*_{t'}$. First (3.28) gives $w^*_{t_2} \cdot \partial w^*_{t_1} \equiv_{R_{acs}} \partial^{n_2} w^*_{t_1} \cdot w^*_{t_2}$, and the rest is the commutativity of

$$\begin{array}{ccccccccc}
\langle I,t'\rangle & \xrightarrow{s_{I_1,I_2}} & \langle I_1,I_2,t'\rangle & \xrightarrow{w^*_{t_1}} & \langle t_1,I_2,t'\rangle & \xrightarrow{\partial w^*_{t_2}} & \langle t_1,t_2,t'\rangle & \xrightarrow{A} & \langle t_1t_2,t'\rangle\\
\Vert & R_{acs}+\text{torsion} & \downarrow s_{n_2,n_1} & (3.30) & \downarrow s_{n_2,1} & (3.29) & \downarrow s_{1,1}=S & (3.12) & \downarrow C_0\\
\langle I,t'\rangle & \xrightarrow{s_{I_2,I_1}} & \langle I_2,I_1,t'\rangle & \xrightarrow{\partial^{n_2}w^*_{t_1}} & \langle I_2,t_1,t'\rangle & \xrightarrow{w^*_{t_2}} & \langle t_2,t_1,t'\rangle & \xrightarrow{A} & \langle t_2t_1,t'\rangle
\end{array}$$

The induction is now easy, and there is no need to consider the case of A_α and C_α separately. So we use X_α to represent the two cases simultaneously. Assume $\alpha = 0\beta$. Then we have $t' = t'_1 t_2$ with $t'_1 = t_1 \bullet X_\alpha$, and we find

$$\begin{aligned}
w_{t'} = c_{I_1,I_2} \cdot w^*_{t'_1} \cdot \partial w_{t_2} &\equiv_{(IH)} c_{I_1,I_2} \cdot w^*_{t_1} \cdot X_{0\beta} \cdot \partial w_{t_2}\\
&\equiv_{\Box} c_{I_1,I_2} \cdot w^*_{t_1} \cdot \partial w_{t_2} \cdot X_{0\beta} = w_t \cdot X_\alpha,\\
w^*_{t'} = s_{I_1,I_2} \cdot w^*_{t'_1} \cdot \partial w^*_{t_2} \cdot A &\equiv_{(IH)} s_{I_1,I_2} \cdot w^*_{t_1} \cdot X_{0\beta} \cdot \partial w^*_{t_2} \cdot A\\
&\equiv_{\Box} s_{I_1,I_2} \cdot w^*_{t_1} \cdot \partial w_{t_2} \cdot X_{0\beta} \cdot A\\
&\equiv_{\Box} s_{I_1,I_2} \cdot w^*_{t_1} \cdot \partial w_{t_2} \cdot A \cdot X_{00\beta} = w^*_t \cdot X_{0\alpha}.
\end{aligned}$$

The argument is symmetric (and simpler: no commutation is needed) in the case $\alpha = 1\beta$. □

Applying Proposition 1.4, we obtain

Proposition 3.17. *The relations R_{ACS} completed with the torsion relations $C_\alpha^2 = S_\alpha^2 = 1$, make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, i.e., of Thompson's group V, in terms of the generators A_α, C_α and S_α.*

As the relations R_{ACS} follow from those of R_{AC} and the definition of S_α, we immediately deduce:

Proposition 3.18. *The relations R_{AC}, i.e., the geometric relations, completed with the pentagon and hexagon relations, and the torsion relations $C_\alpha^2 = 1$, make a presentation of V in terms of the generators A_α and C_α.*

As in the case of the group F, we can restrict to the generators a_i, c_i and s_i. By looking at the proof of Lemma 3.16, we see that, if $t' = t \bullet x_i$ holds with x is a, c, or s, then we have

$$w_{t'} \equiv_{R_{acs}} w_t \cdot x_i. \tag{3.40}$$

Applying Proposition 1.4 once more, we deduce

Proposition 3.19. *The relations R_{acs} completed with the torsion relations $c_i^2 = s_i^2 = 1$ make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, i.e., of V, in terms of the generators a_i, c_i and s_i.*

Finally, as all relations in R_{acs} follow from R_{ac}, we also obtain

Proposition 3.20. *The relations R_{ac} completed with the torsion relations $c_i^2 = 1$ make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, i.e., of V, in terms of the generators a_i and c_i.*

4. Semi-commutativity and the group $\mathfrak{S}_\bullet$

We have seen how to naturally connect Thompson's group V with the associativity and commutativity laws. Inspecting the computations of Section 3, we see that the main technical role is played by the elements S_α. This suggests to introduce the subgroup of $G(\mathcal{A}, \mathcal{C})$ generated by the elements A_α and S_α. We shall see now that the latter naturally arises as a geometry group, namely that of associativity together with a weak form of commutativity.

4.1. The semi-commutativity law.

Definition. We define *(left) semi-commutativity* to be the law

$$x(yz) = (yx)z. \tag{$\mathcal{S}$}$$

As associativity and semi-commutativity are linear laws in the sense of Section 3, they give rise to a geometry group $G(\mathcal{A}, \mathcal{S})$.

Proposition 4.1. *The group $G(\mathcal{A}, \mathcal{S})$ is (isomorphic to) the subgroup $\mathfrak{S}_\bullet$ of V generated by the elements A_α and S_α, i.e., $\mathfrak{S}_\bullet$ is the geometry group of associativity and semi-commutativity.*

Proof. Figure 11 shows that the operators associated with the semi-commutativity law are the operators S_α of Section 3, so the geometry monoid $\mathcal{G}(\mathcal{A}, \mathcal{S})$ is the submonoid of $\mathcal{G}(\mathcal{A}, \mathcal{C})$ generated by the operators $A_\alpha^{\pm 1}$ and $S_\alpha^{\pm 1}$. Quotienting under near-equality gives a similar relation for the geometry groups. □

So, in particular, if we extract from the relations established for $G(\mathcal{A}, \mathcal{C})$ those that involve the generators A_α and S_α only, the latter have to be satisfied in the group $G(\mathcal{A}, \mathcal{S})$.

Definition. We define R_{AS} to consist of the translated copies of

$$X_{0\alpha} \cdot Y_{1\beta} = Y_{1\beta} \cdot X_{0\alpha}, \tag{$\square_\perp$}$$

$$X_{11\alpha} \cdot A = A \cdot X_{1\alpha}, \qquad X_{10\alpha} \cdot A = A \cdot X_{01\alpha}, \qquad X_{0\alpha} \cdot A = A \cdot X_{00\alpha}, \tag{$\square_A$}$$

$$X_{11\alpha} \cdot S = S \cdot X_{11\alpha}, \qquad X_{10\alpha} \cdot S = S \cdot X_{0\alpha}, \qquad X_{0\alpha} \cdot S = S \cdot X_{10\alpha}, \tag{$\square_S$}$$

with $X, Y = A, S$, plus the translated copies of

$$AA = A_1 A A_0, \tag{$\pentagon$}$$

$$SA_1A = A_1AS_0, \quad S_1SA_1 = AS, \quad SS_1A = A_1S, \quad SS_1S = S_1SS_1. \tag{4.1}$$

Proposition 4.2. *All relations of R_{AS}, as well as $S_\alpha^2 = 1$, are satisfied in $G(\mathcal{A}, \mathcal{S})$, i.e., in $\mathfrak{S}_\bullet$.*

We also consider the subfamily of R_{AS} associated with the elements of $\boldsymbol{a}$ and $\boldsymbol{s}$.

Definition. We define R_{as} to consist of the following relations:

$$a_i x_{j-1} = x_j a_i \quad \text{and} \quad s_i x_j = x_j s_i \quad \text{for } j \geqslant i+2 \text{ and } x = a \text{ or } s,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_{i+1} s_i a_{i+1} = a_i s_i, \qquad s_i s_{i+1} a_i = a_{i+1} s_i.$$

Lemma 4.3. *All relations of R_{as} are satisfied in $G(\mathcal{A}, \mathcal{S})$, i.e., in $\mathfrak{S}_\bullet$.*

Actually, it is easy to check that the relations of R_{as} follow from those of R_{AS} plus the definitions $s_i = S_{1^{i-1}}$.

4.2. **Presentations of $\mathfrak{S}_\bullet$.** Our aim is to prove:

Proposition 4.4. *The family R_{AS} completed with the torsion relations $S_\alpha^2 = 1$ make a presentation of $\mathfrak{S}_\bullet$, in terms of the generators A_α and S_α.*

Proof. The method should be clear: we select a family of trees containing one element in each $\mathfrak{S}_\bullet$-orbit, then define distinguished words in $W(\boldsymbol{A}, \boldsymbol{S})$ describing how to construct a tree starting from the distinguished element of its orbit, and, finally, check that there are enough relations in R_{AS} to witness for the relations (1.3) of Proposition 1.4.

The construction is a slight modification of the one used in Section 3. The difference between commutativity and semi-commutativity is that the latter cannot change the rightmost label of a tree. To keep the same conventions as in Section 3, let $T'_{\mathbf{N}}$ denote the subset of $T_{\mathbf{N}}$ made by coloured trees in which the rightmost leaf wears the maximal label. Then every tree in $T'_{\mathbf{N}}$ is equivalent up to associativity and semi-commutativity to some right vine $\langle I \rangle$. For such a tree t, the word w_t maps $\langle I \rangle$ to t, and, by construction, w_t consists of letters a_i and s_j exclusively, since the rightmost leaf is never changed. Indeed, the only letter c_i possibly occurring in w_t comes from the factors $c_{I,J}$ in the inductive construction, and this happens only when I contains the largest element of $I \cup J$. We can therefore use the words w_t and w_t^* without change. Then the only point is to check that $t' = t \bullet X_\alpha$ implies

$$w_{t'} \equiv_{R_{as}} w_t \cdot X_\alpha \quad \text{and} \quad w_{t'}^* \equiv_{R_{as}} w_t^* \cdot X_{0\alpha} \tag{4.2}$$

both in the case $X = A$ and $X = S$. For the case of A_α, it suffices to look at the proof of Lemma 3.16. The case of S has not been considered in Section 3, and we consider it now. So we assume $t = t_1(t_2 t_3)$ and $t' = t_2(t_1 t_3)$. We obtain

$$w_{t'} = s_{I_2, I_1 \cup I_3} \cdot w_{t_2}^* \cdot \partial s_{I_1, I_3} \cdot \partial w_{t_1}^* \cdot \partial^2 w_{t_3}, \tag{4.3}$$

$$w_t \cdot S = s_{I_1, I_2 \cup I_3} \cdot w_{t_1}^* \cdot \partial s_{I_2, I_3} \cdot \partial w_{t_2}^* \cdot \partial^2 w_{t_3} \cdot S. \tag{4.4}$$

By (3.28), we have $w_{t_2}^* \cdot \partial s_{I_1, I_3} \equiv_{R_{as}} \partial^{n_2} s_{I_1, I_3} \cdot w_{t_2}^*$, $w_{t_1}^* \cdot \partial s_{I_2, I_3} \equiv_{R_{as}} \partial^{n_1} s_{I_2, I_3} \cdot w_{t_1}^*$, and $w_{t_2}^* \cdot \partial w_{t_1} \equiv_{R_{as}} \partial^{n_2} w_{t_1} \cdot w_{t_2}^*$. Then the R_{AS}-equivalence of $w_{t'}$ and $w_t \cdot S$ follows from the commutativity of the diagram

$$\begin{array}{ccccccccc}
\langle I \rangle & \xrightarrow{s_{I_1, I_2 \cup I_3} \cdot \partial^{n_1} s_{I_2, I_3}} & \langle I_1, I_2, I_3 \rangle & \xrightarrow{w_{t_1}^*} & \langle t_1, I_2, I_3 \rangle & \xrightarrow{\partial w_{t_2}^*} & \langle t_1, t_2, I_3 \rangle & \xrightarrow{\partial^2 w_{t_3}^*} & \langle t_1, t_2, t_3 \rangle \\
\Vert & R_{as}{+}torsion & \downarrow{\scriptstyle s_{n_2, n_1}} & (3.30) & \downarrow{\scriptstyle s_{n_2, 1}} & (3.29) & \downarrow{\scriptstyle s_{1,1}} & (4.1) & \downarrow{\scriptstyle s_{1,1} = S} \\
\langle I \rangle & \xrightarrow{s_{I_2, I_1 \cup I_3} \cdot \partial^{n_2} s_{I_1, I_3}} & \langle I_2, I_1, I_3 \rangle & \xrightarrow{\partial^{n_2} w_{t_1}^*} & \langle I_2, t_1, I_3 \rangle & \xrightarrow{w_{t_2}^*} & \langle t_2, t_1, I_3 \rangle & \xrightarrow{\partial^2 w_{t_3}^*} & \langle t_2, t_1, t_3 \rangle
\end{array}$$

The relations of R_{AS} are sufficient to obtain the commutativity of the last three squares. As for the first square, the associated permutations are equal, so the relations of R_{as} completed with the torsion relations $s_i^2 = 1$ must give the result.

The argument is similar for the words w_t^*, with an associated diagram coinciding with the above one up to an additional square on the right whose commutativity is provided by the relation $SA_1A = A_1AS_0$. The induction along addresses is similar to the one we used for the groups $G(\mathcal{A})$ and $G(\mathcal{A}, \mathcal{C})$, *i.e.*, for F and V. □

As in Section 2 and 3, we deduce that there are enough relations in the list R_{as} to generate all needed equivalences, and we conclude:

Proposition 4.5. *The group $\mathfrak{S}_\bullet$ is generated by $\boldsymbol{a}$ and $\boldsymbol{s}$, and the relations R_{as} completed with $s_i^2 = 1$ make a presentation of $\mathfrak{S}_\bullet$ in terms of these generators.*

Corollary 4.6. *The group $\mathfrak{S}_\bullet$ is isomorphic to the group $\widehat{V}$ of* [3].

5. The group $B_\bullet$ and its connection to twisted semi-commutativity

The presentation of the group $\mathfrak{S}_\bullet$ in terms of the a_i's and the s_i's given in Proposition 4.5 includes the Coxeter presentation of the symmetric group $\mathfrak{S}_\infty$ in terms of the s_i's. Following the example of Artin's braid group B_∞, which can be defined by removing the torsion relations $s_i^2 = 1$ in the Coxeter presentation of $\mathfrak{S}_\infty$, or, more generally, of Artin–Tits groups, we introduce the group obtained from $\mathfrak{S}_\bullet$ by removing the torsion relations. This is specially natural as we can see that the torsion relations play a very small role in the computations of the previous sections. This new group, here denoted $B_\bullet$, has rich properties, investigated in [9] and [2, 3]. In this paper, we study $B_\bullet$ from the point of view of geometry groups only. The main result is that $B_\bullet$ is the geometry group of associativity together with some twisted version of semi-commutativity. This in particular provides a concrete realization of $B_\bullet$ as a group of partial operators on coloured trees.

5.1. **The group $B_\bullet$.** As is usual with permutations and braids, we use σ_i for the torsion free lifting of the generator s_i. Accordingly, we use $\boldsymbol{\sigma}$ for the infinite family $\sigma_1, \sigma_2, \ldots$, and $R_{a\sigma}$ for a copy of R_{as} with σ_i replacing s_i everywhere.

Definition. We define $B_\bullet$ to be the group $\langle \boldsymbol{a}, \boldsymbol{\sigma}\,; R_{a\sigma}\rangle$, *i.e.*, the group generated by two infinite sequences $a_1, a_2, \ldots, \sigma_1, \sigma_2, \ldots$ with the relations

$$\begin{cases} a_i x_{j-1} = x_j a_i \quad \text{and} \quad \sigma_i x_j = x_j \sigma_i \quad \text{for } j \geqslant i+2 \text{ and } x = a \text{ or } \sigma, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \sigma_{i+1}\sigma_i a_{i+1} = a_i \sigma_i, \qquad \sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i. \end{cases} \tag{5.1}$$

Our current notation is chosen to emphasize the similarity between $B_\bullet$ and Artin's braid group B_∞: as shown in [9], the elements of $B_\bullet$ admit a natural realization in terms of parenthesized braid diagrams, which are analogous to ordinary braid diagrams but with non-uniform distances between the strands. In this framework, σ_i correspond to a standard crossing, while a_i corresponds to a rescaling operator that shrinks the distances around the ith position. The explicit presentation also shows:

Proposition 5.1. *The group $B_\bullet$ is isomorphic to the group $\widehat{BV}$ of* [3].

The group $B_\bullet$ is a sort of twisted product of Thompson's group F and Artin's braid group B_∞, and it is not surprising that it can be investigated by the same methods as F and B_∞. In particular, $B_\bullet$ is a group of left fractions for the monoid with the same presentation [2, 9] and, as least left common multiples exist in this monoid, the group $B_\bullet$ is torsion free.

5.2. **Twisted commutation and semi-commutation.** We turn to the realization of $B_\bullet$ as a geometry group, as we did for F, V, and $\mathfrak{S}_\bullet$. Applying (semi)-commutativity is an involutive operation, while $B_\bullet$ is torsion-free. So we are led to considering non-involutive variants of (semi)-commutativity. A natural way for making commutativity operators non-involutive is to assume that subtrees are changed when they are switched. The simplest case is when only one subtree is changed, and the new subtree depends on the two subtrees that have been exchanged only. This amounts to assuming that there exists a binary operation on trees.

Definition. (Figure 12) Assume that T is a set of trees equipped with a binary operation -[-]. Then we define the *T-twisted commutation* operator C^T by

$$C^T : t_1 \cdot t_2 \longmapsto t_1[t_2] \cdot t_1. \tag{5.2}$$

So we still switch the left and the right subtrees but, in the transformation, the right subtree is (possibly) changed when it crosses the left subtree. The bracket notation is chosen to emphasize that $t_1[t_2]$ is the image of t_2 under the action of t_1. Note that the standard commutation operator C corresponds to using the trivial operation $t_1[t_2] = t_2$.

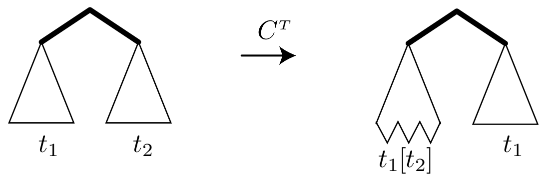


FIGURE 12. The twisted commutation operator C^T

As in the case of the operators C_α, we define C^T_α to be the translated operator $\partial_\alpha C^T$, *i.e.*, C^T acting on the α-subtree. As for inverses, the operators C^T_α need not be injective in general, but we have the following criterion:

Lemma 5.2. *Assume that T is a set of trees equipped with a bracket operation. Then the operators C^T_α are injective if and only the bracket on T is left cancellative, i.e.,*

$$t[t_1] = t[t_2] \quad \textit{implies} \quad t_1 = t_2. \tag{5.3}$$

Under such an hypothesis, the inverse operator of C^T_α is still a partial operator on T.

As we chosed to investigate the torsion-free version $B_\bullet$ of $\mathfrak{S}_\bullet$ rather than that of V, we are led to considering a twisted version of semi-commutation too. We keep the definition of Section 3, *i.e.*, we define the twisted version S^T of S by $S^T = C^T A^{-1} (C^T_1)^{-1}$, which corresponds to:

Definition. (Figure 13) Assume that T is a set of trees equipped with a binary operation -[-]. Then we define the T*-twisted semi-commutation* operator S^T by

$$S^T : t_1 \cdot (t_2 \cdot t_3) \longmapsto t_1[t_2] \cdot (t_1 \cdot t_3). \tag{5.4}$$

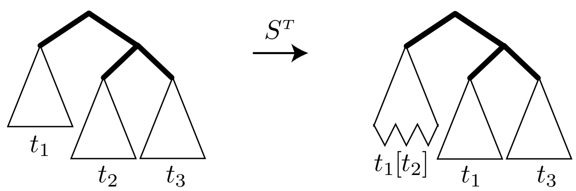


FIGURE 13. The twisted semi-commutation operator S^T

We naturally define S^T_α to be the α-translated copy of S^T. Under the hypothesis that the bracket on T is left cancellative, the operator S^T_α is injective, and its inverse $(S^T_\alpha)^{-1}$ is a partial operator. The (semi)-commutation operators correspond to no algebraic law, but we still have a family of partial self-injections of a set of trees, and it is natural to consider the monoids they generate:

Definition. Assume that T is a family of trees equipped with a left cancellative bracket operation. Then we define $\mathcal{G}(\mathcal{A}, \mathcal{C}^T)$ (*resp.* $\mathcal{G}(\mathcal{A}, \mathcal{S}^T)$) to be the monoid generated by the operators $A^{\pm 1}_\alpha$ and $C^{T\,\pm 1}_\alpha$ (*resp.* the operators $A^{\pm 1}_\alpha$ and $S^{T\,\pm 1}_\alpha$) acting on T.

Our aim is now to investigate the monoids ($\mathcal{G}(\mathcal{A}, \mathcal{C}^T)$ and) $\mathcal{G}(\mathcal{A}, \mathcal{S}^T)$ for appropriate choices of the bracket operation. When T is equipped with the trivial bracket $t_1[t_2] = t_2$, we find

$$\mathcal{G}(\mathcal{A}, \mathcal{C}^T) = \mathcal{G}(\mathcal{A}, \mathcal{C}) \quad \text{and} \quad \mathcal{G}(\mathcal{A}, \mathcal{S}^T) = \mathcal{G}(\mathcal{A}, \mathcal{S}),$$

i.e., we come back to the framework of Sections 3 and 4.

5.3. **LD-systems.** In general, the twisted operators C^T_α and S^T_α need not satisfy the same relations as their standard versions. However it is easy to list the requirements needed for the relations of R_{ACS} to be valid in the monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^T)$.

Proposition 5.3. *(i) The relations $A_1 S^T = S^T S^T_1 A$, $AS^T = S^T_1 S^T A_1$, and $S^T S^T_1 S^T = S^T_1 S^T S^T_1$ hold in the monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^T)$ if and only if, for all trees t_1, t_2, t_3 in T, we have*

$$t_1[t_2 t_3] = t_1[t_2] \cdot t_1[t_2], \tag{5.5}$$

$$(t_1 t_2)[t_3] = t_1[t_2[t_3]], \tag{5.6}$$

$$t_1[t_2[t_3]] = t_1[t_2][t_1[t_3]]. \tag{5.7}$$

(ii) Assume that T is the set of all L-coloured trees for some set L. Then the conditions of (i) are satisfied if and only if there exists a left cancellative left self-distributive bracket operation on L such that, for all trees t_1, t_2 in T_L, the tree $t_1[t_2]$ is obtained by replacing each label y in t_2 with $x_1[x_2[\ldots x_n[y]\ldots]]$, where $(x_1, \ldots, x_n)$ is the left-to-right enumeration of the labels in t_1.

(iii) In this case, all relations of R_{ACS} are satisfied by the operators A_α, C^T_α, and S^T_α, and the torsion relations ${C^T_\alpha}^2 \approx {S^T_\alpha}^2 \approx \mathrm{id}$ are satisfied if and only if, for all trees t_1, t_2, we have

$$t_1[t_1[t_2]] = t_2. \tag{5.8}$$

Proof. For (i), the verifications are given in Figures 14, 15, and 16, respectively. Then (ii) follows from an induction on the size of the trees t_1 and t_2. Finally, in order to establish (iii), it suffices to check the C-geometric relations and the hexagon relations, which is done in Figures 17 and 18. □

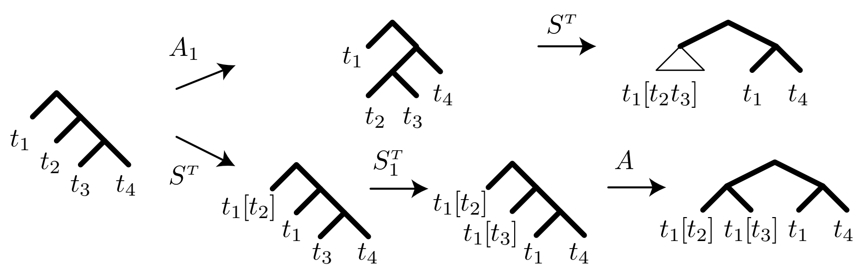


FIGURE 14. The relation $A_1 S^T = S^T S^T_1 A$ requires $t_1[t_2 t_3] = t_1[t_2] \cdot t_1[t_2]$

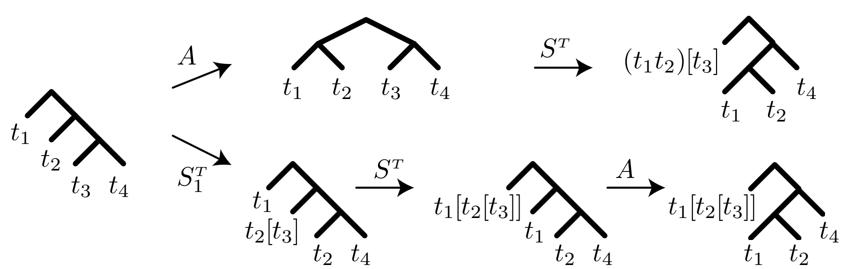


FIGURE 15. The relation $AS^T = S^T_1 S^T A_1$ requires $(t_1 t_2)[t_3] = t_1[t_2[t_3]]$

Remark 5.4. As we are mostly interested in the group $B_\bullet$, we concentrated on the constraints guaranteeing that the relations of R_{AS} are satisfied, and we saw that all relations of R_{ACS} are then valid. If we start with the operators C^T_α and require that the relations of R_{AC} be satisfied, we come up with exactly the same constraints, as can be read in Figures 17 and 18.

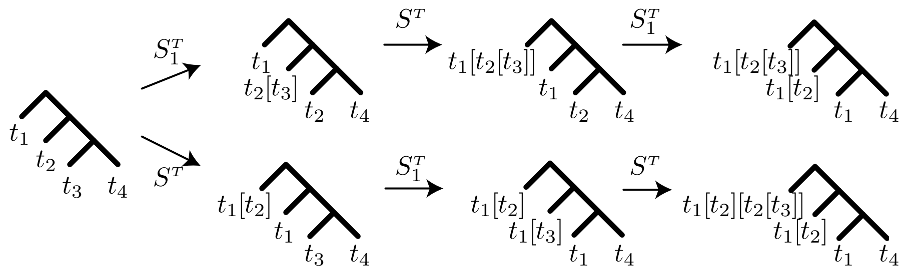


FIGURE 16. The relation $S_1^T S^T S_1^T = S^T S_1^T S^T$ requires $t_1[t_2[t_3]] = t_1[t_2][t_1[t_3]]$

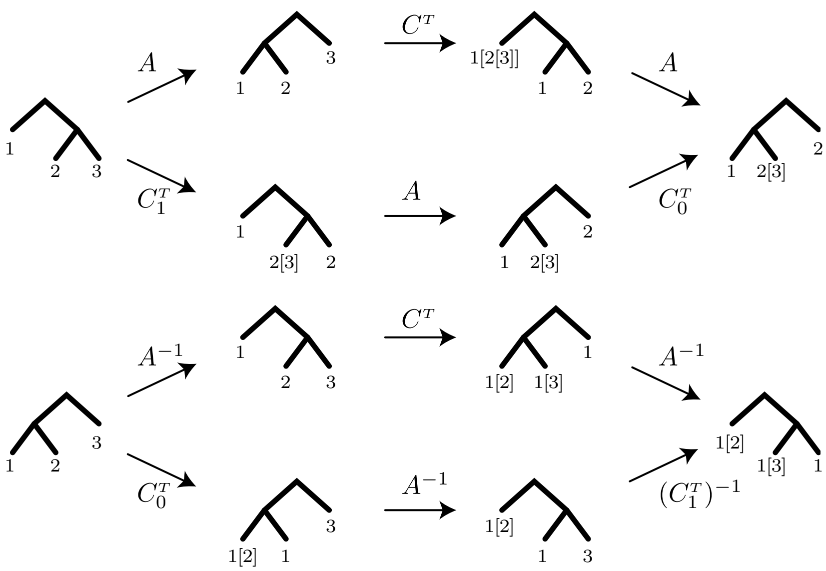


FIGURE 17. The twisted hexagon relations

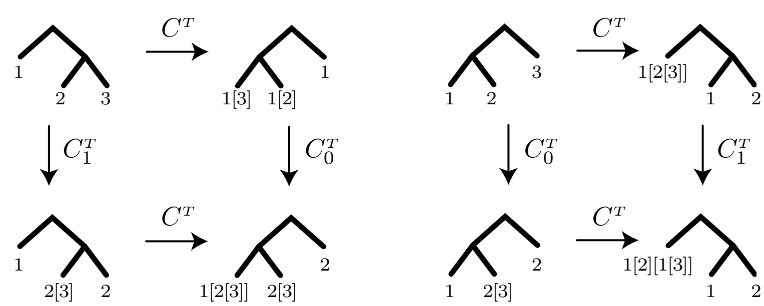


FIGURE 18. The twisted C-geometric relations

We shall therefore be interested in the sequel with sets equipped with a left self-distributive operation, *i.e.*, a binary operation that satisfies the algebraic law

$$(LD) \qquad x[y[z]] = x[y][x[z]]$$

—or $x(yz) = (xy)(xz)$ when the operation symbol is omitted.

Definition. An algebraic system consisting of a set equipped with a left self-distributive operation is called an *LD-system*. An LD-system is said to be left cancellative if its left translations are injective, *i.e.*, if (5.3) holds; it is called an *LD-quasigroup* (in [7]) or a *rack* (in [11]) if its left translations are bijective. An LD-system is said to be *involutory* if (5.8) holds. Note that an involutory LD-system is necessarily an LD-quasigroup.

Example 5.5. Any set L equipped with $x[y] = y$ is a (trivial) involutory LD-system. If G is a group, then G equipped with $x[y] = xyx^{-1}$ is an LD-quasigroup, denoted $\mathrm{conj}(G)$ in the sequel.

From now on, we always restrict to the context of Proposition 5.3(*ii*), *i.e.*, consider the twisted (semi)-commutation operators on the set T_L that stem from some left cancellative LD-system L. Accordingly, we shall simplify our notation, and write $\mathcal{G}(\mathcal{A}, \mathcal{S}^L)$ for $\mathcal{G}(\mathcal{A}, \mathcal{S}^{T_L})$, and, similarly, $\mathcal{G}(\mathcal{A}, \mathcal{C}^L)$ for $\mathcal{G}(\mathcal{A}, \mathcal{C}^{T_L})$.

5.4. **Making groups.** As in the case of associativity and semi-commutativity, and for each fixed left cancellative LD-system L, one can derive a group from the monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^L)$ by identifying near-equal operators. However, controlling a possible collapsing is not trivial, as we are not in the framework of linear algebraic laws.

The problem is to show that the near-equality relation $\approx$ defines a congruence on the monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^L)$. As in Section 3, the solution is to show that each operator admits a convenient seed in order to deduce that $\approx$ is transitive. Now the notions of a substitution and, consequently, of a seed, have to be adapted to our current context. At the expense of considering coloured trees whose labels are formal expressions containing variables and bracket operations, one can show that, in a convenient sense, the pair of coloured trees $(\langle 1, 2\rangle, \langle 1[2], 1\rangle)$, *i.e.*, $(\bullet_1\bullet_2, \bullet_{1[2]}\bullet_1)$, is a seed for the operator C^T, while $(\langle 1, 2, 3\rangle, \langle 1[2], 1, 3\rangle)$ is a a seed for the operator S^T. The details are easy in the case of an LD-quasigroup; in the more general case of a left cancellative LD-system, more care is needed, but all required techniques are explained in Chapter VIII of [7]. The key ingredient is the result that, if two self-distributivity operators (analogous to the current operators A_α or S_α but for the left self-distributivity law) agree on some tree, then they agree everywhere. All we need in the sequel is the following result:

Lemma 5.6. *Assume that L is a left cancellative LD-system. Then near-equality is a congruence on the monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^L)$, and the action of the latter on T_L induces a partial action of the associated quotient-group $G(\mathcal{A}, \mathcal{S}^L)$.*

The group $G(\mathcal{A}, \mathcal{S}^L)$ will naturally be called the geometry group of associativity and L-twisted semi-commutativity. Proposition 5.3 directly implies:

Proposition 5.7. *For each left cancellative LD-system L, the group $G(\mathcal{A}, \mathcal{S}^L)$ is a quotient of $B_\bullet$.*

The group $G(\mathcal{A}, \mathcal{S}^L)$ depends on the considered LD-system L. For instance, when L is any infinite set equipped with the trivial operation $x[y] = y$, then $G(\mathcal{A}, \mathcal{S}^L)$ coincides with $G(\mathcal{A}, \mathcal{S})$, *i.e.*, with $\mathfrak{S}_\bullet$. On the other hand, we can expect that non-trivial LD-systems give rise to larger geometry groups, and we can in particular raise:

Question 5.8. *Does there exist a left cancellative LD-system L satisfying $G(\mathcal{A}, \mathcal{S}^L) = B_\bullet$?*

A positive answer would correspond to what can be called a geometric realization of $B_\bullet$, *i.e.*, a realization of $B_\bullet$ as the geometry group of associativity and twisted semi-commutativity.

5.5. **$B_\bullet$-twisted semi-commutativity.** In order to answer Question 5.8 in the positive, we have to exhibit a convenient LD-system. Several solutions are possible, but the quickest and maybe most interesting one involves a self-distributive structure on $B_\bullet$ itself.

Definition. For x, y in $B_\bullet$, we set

$$x[y] = x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1}, \tag{5.9}$$

$$x \circ y = x \cdot \partial y \cdot a_1. \tag{5.10}$$

Proposition 5.9. *The set $B_\bullet$ equipped with the bracket operation is a left cancellative LD-system. Moreover, the following mixed relations are satisfied*

$$x[y[z]] = (x \circ y)[z], \qquad x[y \circ z] = x[y] \circ x[z], \tag{5.11}$$

where ∂ denotes the endomorphism of $B_\bullet$ that maps σ_i to σ_{i+1} and a_i to a_{i+1} for every i.

The self-distributivity of the bracket opertaion and the relations (5.11) follow from the relations of $R_{a\sigma}$ using easy verifications; proving that the bracket operation is left cancellativity requires to know that the endomorphism ∂ is injective, which in turn uses the decomposition of $B_\bullet$ as a group of fractions. As the arguments appear in [9], we shall not repeat them here.

Proposition 5.10. *The group $G(\mathcal{A}, \mathcal{S}^{B_\bullet})$ is (isomorphic to) $B_\bullet$, i.e., $B_\bullet$ is the geometry group of associativity and $B_\bullet$-twisted semi-commutativity.*

Proving Proposition 5.10 amounts to proving that the relations $R_{a\sigma}$ make a presentation of the group $G(\mathcal{A}, \mathcal{S}^{B_\bullet})$ in terms of the generators a_i and σ_i, *i.e.*, equivalently, that the canonical surjective hommorphism of $B_\bullet$ onto $G(\mathcal{A}, \mathcal{S}^{B_\bullet})$ is an isomorphism. We use Proposition 1.3. To this end, we associate with every $B_\bullet$-coloured tree a distinguished element of $B_\bullet$ in such a way that the (external) action of $B_\bullet$ on trees corresponds to an (internal) multiplication inside $B_\bullet$. We proceed in two steps.

Definition. For t a $B_\bullet$-coloured tree, we define $e(t)$ to be the $\circ$-evaluation of t, *i.e.*, to be the element of $B_\bullet$ inductively defined by $e(\bullet_x) = x$ and $e(t_1 t_2) = e(t_1) \circ e(t_2)$. Then we put

$$f(t) = e(t_1) \cdot \partial e(t_2) \cdot \ldots \cdot \partial^{n-1} e(t_n)$$

where $\langle t_1, \ldots, t_n, \bullet_x \rangle$ is the decomposition of t along its right branch.

The element $f(t)$ is also defined by the inductive rules $f(\bullet_x) = 1$ and $f(t_1 t_2) = e(t_1) \cdot \partial f(t_2)$. Observe that the definitions of $e(t)$ and $f(t)$ are parallel to those of w_t^* and w_t in Section 2. The key point is the following computation:

Lemma 5.11. *Assume that t is an $B_\bullet$-coloured tree. Then we have*

$$f(t \bullet a_i) = f(t) \cdot a_i, \qquad f(t \bullet \sigma_i) = f(t) \cdot \sigma_i, \tag{5.12}$$

whenever the involved trees are defined.

Proof. First, we observe that, for all $B_\bullet$-coloured trees t_1, t_2, we have

$$e(t_1[t_2]) = e(t_1)[e(t_2)], \tag{5.13}$$

as follows from an induction on the sizes of t_1 and t_2, using the relations of (5.11)

Now, for $t = t_1 \ldots t_n \bullet_x$, let $D(t)$ denote the sequence $(t_1, \ldots, t_n)$, and let $E(t)$ be the sequence $(e(t_1), \ldots, e(t_n))$. By definition, we have

$$D(t \bullet a_i) = (t_1, \ldots, t_{i-1}, t_i t_{i+1}, t_{i+2}, \ldots, t_n),$$
$$D(t \bullet \sigma_i) = (t_1, \ldots, t_{i-1}, t_i[t_{i+1}], t_i, t_{i+2}, \ldots, t_n).$$

Hence, assuming $E(t) = (x_1, \ldots, x_n)$, and using (5.13) for the second relation, we obtain

$$E(t \bullet a_i) = (x_1, \ldots, x_{i-1}, x_i \circ x_{i+1}, x_{i+2}, \ldots, x_n),$$
$$E(t \bullet \sigma_i) = (x_1, \ldots, x_{i-1}, x_i[x_{i+1}], x_i, x_{i+2}, \ldots, x_n)$$

whenever the involved terms are defined. Using the explicit definition of $f(t \bullet a_i)$ and $f(t \bullet \sigma_i)$ from $E(t \bullet a_i)$ and $E(t \bullet \sigma_i)$ then easily gives (5.12) using the relations of R_{as}. □

We are now able to conclude.

Proof of Proposition 5.10. We are in position for applying Proposition 1.3. Indeed, we have a surjective homomorphism $B_\bullet \to G(\mathcal{A}, \mathcal{S}^{B_\bullet})$ together with a map $f : T_{B_\bullet} \to B_\bullet$ satisfying (5.12), which are the relations (1.1) corresponding to the generating subset $\boldsymbol{a} \cup \boldsymbol{\sigma}$ of $B_\bullet$. □

5.6. **The group of general twisted semi-commutativity.** To conclude with a simple statement, let $S_\alpha^\#$ denote the union of all operators $S_\alpha^{T_L}$ (considered as sets of pairs) for all possible sets T_L associated with a left cancellative LD-system, and define $\mathcal{G}(\mathcal{A}, \mathcal{S}^\#)$ to be the monoid generated by all operators A_α, $S_\alpha^\#$ and their inverses. By construction, each specific monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^L)$ is a quotient of $\mathcal{G}(\mathcal{A}, \mathcal{S}^\#)$. Then near-equality is still a congruence on $\mathcal{G}(\mathcal{A}, \mathcal{S}^\#)$, and the corresponding group $G(\mathcal{A}, \mathcal{S}^\#)$ naturally appears as the geometry group of associativity and (general) twisted semi-commutativity. We can state:

Proposition 5.12. *The group $G(\mathcal{A}, \mathcal{S}^\#)$ is (isomorphic) to $B_\bullet$, i.e., $B_\bullet$ is the geometry group of associativity and twisted semi-commutativity.*

Proof. By construction, the group $G(\mathcal{A}, \mathcal{S}^{B_\bullet})$ is a quotient of the general group $G(\mathcal{A}, \mathcal{S}^\#)$. By Proposition 5.3(*iii*), the group $G(\mathcal{A}, \mathcal{S}^\#)$ is a quotient of $B_\bullet$. Now, Proposition 5.10 shows that the canonical mapping of $B_\bullet$ to $G(\mathcal{A}, \mathcal{S}^{B_\bullet})$ is an isomorphism, so the two surjective homomorphisms of which the latter is the product must be isomorphisms as well. □

We mentioned above that group conjugacy provides examples of left cancellative LD-systems. Therefore, we obtain for each particular group G a notion of conj(G)-twisted (semi)-commutativity, with an associated inverse monoid $\mathcal{G}(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$ and the associated group $G(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$. The latter group depends on the group G: if G is abelian, conjugacy is trivial on G, and the geometry group $G(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$ is therefore $\mathfrak{S}_\bullet$, as was proved in Section 4. On the other hand, if G is a non-abelian free group, conjugacy is not trivial, and we raise the question of recognizing the corresponding geometry group.

Proposition 5.13. *If G be a free group of rank at least 2, the group $G(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$ is (isomorphic to) $B_\bullet$, i.e., $B_\bullet$ is the geometry group of associativity and* conj(G)*-twisted semi-commutativity.*

Proof (sketch). Without loss of generality, we can assume that G is a free group based on a family of generators x_α indexed by binary addresses, *i.e.*, finite sequences of 0's and 1's. The problem is to show that, if w is a word in $W(\boldsymbol{a}, \boldsymbol{\sigma})$, then the image $\overline{w}$ of w in $B_\bullet$ can be recovered from the operator of $\mathcal{G}(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$ associated with w.

Now, it is shown in [9] that Artin's representation of the braid group B_∞ extends to $B_\bullet$: there exists an injective morphism ψ of $B_\bullet$ into $\mathrm{Aut}(G)$. Hence, it suffices to prove that $\psi(\overline{w})$ is determined by the operator of $\mathcal{G}(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$ associated with w. We claim that there exists a G-coloured tree t such that $t \bullet w$ exists and $\psi(\overline{w})$ can be recovered from the pair $(t, t \bullet w)$, hence *a fortiori* from the operator of $\mathcal{G}(\mathcal{A}, \mathcal{S}^{\mathrm{conj}(G)})$ associated with w.

Let us say that a G-coloured tree t is *natural* if the labels of t of each leaf with address $\alpha 01^k$ is $x_{\alpha 01^{k-1}}^{-1} \dots x_{\alpha 01}^{-1} x_{\alpha 0}^{-1} x_\alpha$ and the one of the leaf with address 1^k is $x_{1^{k-1}}^{-1} \dots x_1^{-1} x_\phi^{-1}$. Proposition 5.4 of [9] shows (with different notation) that, if t is a natural G-coloured tree, and w is a word in $W(\boldsymbol{a}, \boldsymbol{\sigma})$ such that $t \bullet w$ is defined, then, for each address γ such that $\gamma 0$ is the address of a leaf in $t \bullet w$, the image of x_γ under $\psi(\overline{w})$ is the label at $\gamma 0$ in $t \bullet w$: the property can be checked for a_i and σ_i directly, and, then, one uses an induction on the length of w. As we can choose t as large as we wish, this shows that $\psi(\overline{w})$, hence $\overline{w}$, is determined by the action of w on G-coloured trees. □

Proposition 5.13 gives an alternative proof of Proposition 5.12.

As a final remark, let us observe that the above treatment of twisted semi-commutativity and its connection with the group $B_\bullet$ can be repeated for twisted commutativity and its connection with the group obtained by removing the torsion relations $c_i^2 = 1$ in the presentation of V described in Section 3. The latter group is (isomorphic to) the group denoted BV in [2, 3], and it also identifies with the subgroup of $B_\bullet$ generated by the elements $a_1^{-1} \dots a_i^{-1} a_{i+1}^{-1} a_i a_i \dots a_1$ and $a_1^{-1} \dots a_i^{-1} \sigma_i a_i \dots a_1$ corresponding to the elements that, under the action by associativity and twisted semi-commutativity, act trivially outside the 0-subtree.

References

[1] M. Brin, *The algebraic structure of the braided Thompson group*, Preprint (2003).

[2] M. Brin, *The algebra of strand splitting. I. A braided version of Thompson's group V*, arXiv math.GR/040642.

[3] M. Brin, *The algebra of strand splitting. II. A presentation for the braid group on one strand*, arXiv math.GR/040643.

[4] K. Brown & R. Geoghegan, *An infinite-dimensional torsion-free FP_∞ group*, Invent. Math. **77** (1984) 367–381.

[5] J.W. Cannon, W.J. Floyd, & W.R. Parry, *Introductory notes on Richard Thompson's groups*, Ens. Math. **42** (1996) 215–257.

[6] P. Dehornoy, *The structure group for the associativity identity*, J. Pure Appl. Algebra **111** (1996) 59–82.

[7] P. Dehornoy, *Braids and Self-Distributivity*, Progress in Math. vol. 192, Birkhäuser (2000).

[8] P. Dehornoy, *Study of an identity*, Alg. Universalis **48** (2002) 223–248.

[9] P. Dehornoy, *The group of parenthesized braids*, arXiv: math.GR/0407097.

[10] P. Dehornoy, *Lattice structures on Thompson's group F*, in preparation.

[11] R. Fenn & C.P. Rourke, *Racks and links in codimension 2*, J. of Knot Theory and its Ramifications (1992) 343–406;

[12] L. Funar & C. Kapoudjian, *On a universal mapping class group in genus zero*, GAFA; to appear.

[13] F.A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford **20-78** (1969) 235–254.

[14] P. Greenberg & V. Sergiescu, *An acyclic extension of the braid group*, Comment. Mat. Helvetici **66** (1991) 109–138.

[15] J.M. Howie, *An introduction to semigroup theory*, Academic Press (1976).

[16] C. Kapoudjian & V. Sergiescu, *An extension of the Burau representation to a mapping class group associated to Thompson's group T*, Contemp. Math.; to appear.

[17] J. Kellendonck & M. Lawson, *Partial actions of groups*, Inter. J. of Alg. and Computation; to appear.

[18] M. Lawson, *Constructing ordered groupoids*, Preprint (2004).

[19] S. MacLane, *Natural associativity abd commutativity*, Rice University Studies **49** (1963) 28–46.

[20] R. McKenzie & R.J. Thompson, *An elementary construction of unsolvable word problem in group theory*, Word Problems (W. Boone, ed.); North Holland, 1973, pp. 457–478.

[21] M. Megrelishvili & L. Schröder, *Globalization of confluent partial actions on topologicial and metric spaces*, Preprint (2002).

[22] A. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Math. vol. 170, Birkhaüser (1998).

[23] D. Sleator, R. Tarjan, & W. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, J. Amer. Math. Soc. **1-3** (1988) 647–681.

[24] J.D. Stasheff, *Homotopy associativity of H-spaces*, Trans. Amer. Math. Soc. **108** (1963) 275–292.

[25] R.J. Thompson, *Embeddings into finitely generated simple groups which preserve the word problem*, Word Problems II (S. Adian, W. Boone, G. Higman, eds), North Holland, 1980, pp. 401–441.

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