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Preface

The aim of this book is to present recently discovered connections between Artin's braid groups B_n and left self-distributive systems (also called LD-systems), which are sets equipped with a binary operation satisfying the left self-distributivity identity

$$x(yz) = (xy)(xz). \qquad (LD)$$

Such connections appeared in set theory in the 1980s and led to the discovery in 1991 of a left invariant linear order on the braid groups.

Braids and self-distributivity have been studied for a long time. Braid groups were introduced in the 1930s by E. Artin, and they have played an increasing role in mathematics in view of their connection with many fields, such as knot theory, algebraic combinatorics, quantum groups and the Yang–Baxter equation, *etc.* LD-systems have also been considered for several decades: early examples are mentioned in the beginning of the 20th century, and the first general results can be traced back to Belousov in the 1960s. The existence of a connection between braids and left self-distributivity has been observed and used in low dimensional topology for more than twenty years, in particular in work by Joyce, Brieskorn, Kauffman and their students. Brieskorn mentions that the connection is already implicit in (Hurwitz 1891).

The results we shall concentrate on here rely on a new approach developed in the late 1980s and originating from set theory. Most of the examples of self-distributive operations known at the time were connected with conjugacy in a group, and what set theory provided was a new example of a completely different type—together with the hint that something deep was hidden there, and the frustrating situation that the very existence of the example was an unprovable statement. Developing alternative constructions without relying on an unprovable statement has led to a new geometrical approach to self-distributive algebra, which has made the connection with braids more striking, and has led to a number of results about left self-distributivity, in particular many new examples and a complete description of free LD-systems; it has also led to new results about braids, the most promising so far being the existence of a natural linear order on braids.

Our story began around 1983 when the main challenge of set theory was to establish a connection between large cardinal axioms and Projective Determinacy, a structural statement that describes the Lusin hierarchy on the real line. Most large cardinal axioms involve elementary embeddings, which are a kind of endomorphism relevant in the context of set theory, and it was part of the folklore that certain elementary embeddings can be equipped with a left self-distributive operation, resulting in an intricate calculus of iterates. A proof of Projective Determinacy using large cardinals, namely Woodin cardinals, was given by Martin and Steel in 1985, and this proof, as well as a subsequent alternative proof by Woodin, requires much more than computing iterates of elementary embeddings. However, two results of the time (1986) are the author's observation that the existence of the left self-distributive operation on elementary embeddings is sufficient to prove Analytic Determinacy, a nontrivial fragment of Projective Determinacy involving the first two levels of the Lusin hierarchy, and Laver's theorem (which appeared only later) that left division in the LD-systems of elementary embeddings has no cycle. The former result shows that the self-distributive operation is nontrivial in a strong sense; the latter shows that it is quite different from the conjugacy in a group.

Motivated by the previous observations, R. Laver and the author undertook a systematic study both of the LD-systems of elementary embeddings and of free LD-systems. Contrary to related free systems, such as Joyce's free quandles or Fenn and Rourke's free racks, the free LD-systems had not received any attention so far, probably because examples were missing. The most significant result obtained during this period was the existence of a left invariant linear order on free LD-systems of rank 1 and, as a corollary, a solution to the word problem of Identity (LD), under the hypothesis that there existed an LD-system in which left division has no cycle; this was proved by R. Laver and the author independently in the Spring of 1989.

The above results created a strange situation: the existence of a left invariant order on free LD-systems, or the decidability of the word problem of (LD), are effective, finitistic properties; yet, their proof required the existence of an LD-system with a certain property, and the only known examples of such LD-systems were the LD-systems of elementary embeddings. Now, an inevitable consequence of Gödel's incompleteness theorem is that the existence of an elementary embedding of the required type is an unprovable statement, actually a higher infinity axiom. Hence the existence of an LD-system of elementary embeddings is unprovable. Therefore, the logical status of the previous results was unclear, and new developments were needed, either to construct alternative examples of LD-systems resorting to no set theoretical axiom, or to prove that such axioms were needed, as is known to be in the case of Projective Determinacy.

Here braids came into the picture, indirectly at first. The main problem was to study free LD-systems. As no concrete realization was available, the only possible approach was a syntactic approach. The main idea here has been to introduce a certain group G_{LD} which captures some geometrical properties of the identity (LD). The group G_{LD} happens to be an extension of the union B_∞ of all Artin's braid groups B_n, and it has been investigated using algebraic tools analogous to the ones Garside used for braid groups. At the end of 1991, this study led to a proof—without any set theoretical hypothesis—that left division in free LD-systems has no cycle and, as a natural by-product, to the construction of a left self-distributive operation and of a left invariant linear order on braids.

Further results both about braid ordering and about LD-systems were then proved by several authors. First, once the explicit definition of the above-mentioned self-distributive operation on braids became available, a shorter proof for existence of the braid ordering could be given, as was noted by Larue in 1992. In 1993, Laver proved that the restriction of the order to n-strand positive braids is a well-ordering. In 1995, we derived from the braid ordering a new efficient method for recognizing braid isotopy. In 1997, Fenn, Greene, Rolfsen, Rourke, and Wiest reconstructed the braid order by purely topological means, and extended the method to the mapping class group of any surface with a nonempty boundary. More recently, Short and Wiest, building on a suggestion by Thurston and work by Nielsen, gave another definition involving hyperbolic geometry and an action of braids on the real line.

As for self-distributive algebra, *i.e.*, the study of LD-systems, the main developments involve the free LD-systems, the LD-systems of elementary embeddings, and some finite LD-systems which we call the Laver tables. The results about free LD-systems consist mainly in the construction of several normal forms by Laver and the author (around 1992), and in a deepening of our understanding of the group G_{LD} (recent work). About the LD-systems of elementary embeddings, a number of results were proved by Laver, Dougherty, and Jech between 1988 and 1995 in connection with the computation of the so-called critical ordinals and attempts to construct finitistic counterparts that do not rely on large cardinal axioms. The Laver tables were introduced by Laver in the 1980s as finite quotients of the LD-systems of elementary embeddings. They are fascinating objects with a formidable combinatorial complexity. Many results about them were proved in the 1990s by Drápal. A remarkable point is that, contrary to the existence of an LD-system with an acyclic left division, some results about the Laver tables, proved using elementary embeddings, have not yet received alternative combinatorial proofs, and, therefore, they still depend on an unprovable logical axiom.

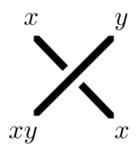
Looking at the current picture, we see that set theory is not involved in any of the braid constructions, and one may feel that its involvement in the results about the Laver tables simply implies that our understanding of these objects is far from complete. However, we would argue that both the braid order and the Laver tables are, and are to remain, applications of set theory: if the latter had not clearly shown the way, it is more than likely that most of the results we shall present in this monograph would not have been discovered yet. Similarly, the existence of the braid order can now be established quickly using Larue's short proof that left division associated with the left self-distributive operation on braids has no cycle: yet, it is not clear that this operation would have been discovered without a study of the group G_{LD}.

About this book

The current text proposes a first synthesis of this area of research. Our exposition is self-contained, and there are no prerequisites. This leads us to establish a number of basic results, about braids, self-distributive algebra, and, to some extent, set theory. However, the text is not a comprehensive course about these subjects, as we have selected only those results that are needed for our specific, mainly geometry–oriented, approach.

The text is divided into three parts, devoted to the braid order, to free LD-systems, and to general LD-systems respectively. This order does not follow the chronology of development, but it gives the braid applications first in order to motivate the study of free LD-systems that comes next. Set theory is postponed to the end, when we cannot avoid it any longer. The three parts are rather independent, and it is possible to begin with any of them, at least for a first glance.

Let us give a preview of each part of the book. The aim of Part A is to construct the linear order of braids and establish its main properties. The point is the existence of an action of braids on powers of LD-systems: if S is a given LD-system, and b is an n-strand braid, then, for each element $\vec{a}$ in S^n, we can define a new element $\vec{a} \bullet b$ by seeing $\vec{a}$ as a set of colours put on the top of b and $\vec{a} \bullet b$ as the resulting set of colours at the bottom of b when the colours flow down b according to the rule



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When one uses classical examples of LD-systems, such as the conjugacy in a group or the barycentric mean, one deduces well-known results, such as the representation of braids inside automorphisms of a free group, and the Burau representation. New results appear when we use non-classical examples of LD-systems. In particular, a linear ordering of braids appears naturally when we use a linearly ordered LD-system $(S, <)$, *i.e.*, when the colours are ordered: the obvious idea is to extend the order to S^n lexicographically, and to define the braid b_1 to be smaller than the braid b_2, if $\vec{a} \bullet b_1 < \vec{a} \bullet b_2$ holds for every element $\vec{a}$ of S^n. The required compatibility condition turns out to be that the linear order on S satisfies $a < ab$ for all a, b, and the main technical result that makes the construction possible is that the free LD-system of rank 1 admits such a linear ordering. Technically, we avoid using abstract free LD-systems here and we resort instead to a self-distributive operation defined on braids.

Part A comprises four chapters. In Chapter I, we give an introduction to braids, and we present quickly (and somewhat informally) the connections between braids and self-distributivity, namely the above-mentioned action of braids on LD-systems, and the existence of a left self-distributive operation (called exponentiation) on braids. We also introduce what we call extended braids.

Chapter II is a preparatory chapter in which we develop a specific combinatorial method for studying those groups and monoids with a presentation of a particular syntactic form. We show that the braid groups are eligible, and deduce a number of classical properties of these groups. The method is used in Chapter VIII again.

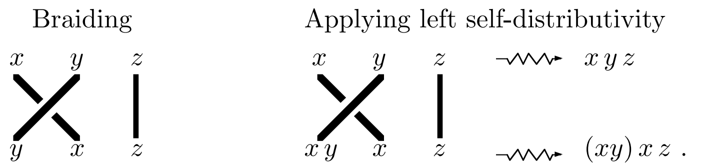
Chapter III is devoted to the linear order of braids. Using the word reversing technique of Chapter II, we extend the action of braids on powers of LD-systems defined in Chapter I to more general LD-systems. Admitting a general result about monogenic LD-systems that will be proved in Chapter V, we deduce the existence of a left invariant order on braids. We also mention alternative definitions of this order.

In Chapter IV, we study the restriction of the braid order to positive braids, *i.e.*, to those braids that can be expressed without using the inverses σ_i^{-1} of the standard braid group generators. The main result is Laver's theorem that the restriction of the order to n-strand positive braids is a well-ordering. Here we present Burckel's construction, which computes the order type effectively and provides a new normal form for positive braids.

Part B is a general study of free LD-systems. Should the associativity identity replace left self-distributivity, the free systems would be the free semigroups, hence rather simple objects. Free LD-systems are actually much more complicated, yet they share many properties with free semigroups, and, in particular, the property that the free system of rank 1 is equipped with a unique linear order such that $a < ab$ always holds: in the case of associativity, the free semigroup of rank 1 is the set of positive integers, and the corresponding linear order is the usual order of the integers. The core of our study consists in investigating LD-equivalence of terms. The latter are abstract expressions (or words) involving variables, one binary operator, and parentheses, and we say that two terms t, t' are LD-equivalent if we can transform t into t' by applying Identity (LD) once or several times. If we think of associativity again, applying the identity to a term amounts to changing the place of parentheses, so that every equivalence class is finite, and we can select a unique representative easily, for instance by pushing all parentheses to the right. The case of LD-equivalence is much more complicated: in general, the equivalence class of a term is infinite, and finding distinguished representatives is a serious task. The main point in the approach we develop here is to put the emphasis on geometrical features*. We introduce a certain monoid $\mathcal{G}_{LD}$ consisting of partial mappings on terms, and a related group G_{LD}, so that the LD-equivalence class of a term t becomes its orbit under some partial action of G_{LD}**. All results about braids mentioned in Part A then originate from results on G_{LD}: the existence of the braid action on LD-systems comes from Artin's braid group B_∞ being a quotient of G_{LD}; braid exponentiation originates from expressing in G_{LD} a simple general property of left self-distributivity called absorption; the linear ordering of braids comes from some linear preordering on G_{LD}. The situation can be summarized in the slogan

> *The geometry of left self-distributivity is a refinement of the geometry of braids,*

which is reminiscent of the well-known observation that the geometry of braids is a refinement of the geometry of permutations: going from the symmetric group $\mathfrak{S}_\infty$ to the braid group B_∞ amounts to completing a permutation with some information about how the transpositions have been performed; similarly, going from B_∞ to G_{LD} amounts to completing a braid with some additional information about the names of the strands that have been braided, as the diagram below may suggest:



Part B is divided into five chapters. In Chapter V, we introduce free LD-systems and develop a convenient framework of finite trees. We prove the Comparison Property, which is the missing fragment in the construction of the braid order in Chapter III, and the existence of left invariant linear orders on free LD-systems, which implies the decidability of the word problem.

In Chapter VI, we prove unique normal form results for LD-equivalence, *i.e.*, we construct families of distinguished terms so that every term is LD-equivalent to exactly one distinguished term. This requires developing a precise analysis of the geometry of terms. Various applications are given, in particular about braids and their exponentiation.

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* R. Laver has developed an alternative approach, briefly mentioned at the end of Chapter VI; neither approach is subsumed in the other, as both prove statements seemingly inaccessible to the other.

** A similar approach can be developed for every identity: in the case of associativity, the corresponding group is Richard Thompson's group F.

In Chapter VII, we introduce the monoid $\mathcal{G}_{LD}$ by considering partial operators acting on terms and specifying how and where left self-distributivity is applied. We exhibit a family of relations, called LD-relations, which hold in $\mathcal{G}_{LD}$, reminiscent of those relations involved in the MacLane–Stasheff pentagon in the case of associativity. We check that most of the results about free LD-systems established in Chapter V can be deduced from LD-relations.

In Chapter VIII, we introduce the group G_{LD} for which LD-relations yield a presentation. We describe the connection between $\mathcal{G}_{LD}$ and G_{LD}, and we construct counterparts to the braid exponentiation and to the braid order in G_{LD}. In particular, we give a purely syntactical proof of the fundamental result that left division in a free LD-system admits no cycle.

In Chapter IX, we deepen our study of the group G_{LD} and of the associated positive monoid M_{LD}. By refining the results of Chapter VI, we prove partial results about the Polish Algorithm and the Embedding Conjecture: The Polish Algorithm is a natural syntactic method for deciding LD-equivalence of terms, and its termination is one of the most puzzling open questions of the subject. The Embedding Conjecture claims that the monoid M_{LD} embeds in the group G_{LD}, a counterpart to Garside's result that the braid monoid B_n^+ embeds in the group B_n. This conjecture is equivalent to a number of properties of left self-distributivity.

Part C contains further developments about LD-systems. We did not try to be exhaustive, and we concentrated on some special families of LD-systems, in particular the Laver tables and the LD-systems of elementary embeddings. The Laver tables form an infinite family A_0, A_1, $\ldots$ of finite monogenic LD-systems. The LD-system A_n has 2^n elements, and it is the unique LD-system with domain $\{1, 2, \ldots, 2^n\}$ satisfying $a*1 = a+1 \bmod 2^n$ for every a; projection *modulo* 2^n defines a surjective homomorphism of A_{n+1} onto A_n for every n, so the family $(A_n)_n$ is an inverse system. On the other hand, ranks are special sets with the weird property that every mapping of R into itself can be seen as an element of R, and elementary embeddings are defined as nontrivial injective mappings that preserve every notion that is definable from the membership relation $\in$. Let us say that a rank R is self-similar if there exists an elementary embedding of R into itself which is not the identity. The point is that, if i and j are elementary embeddings of some self-similar rank R into itself, then, as i applies to every element of R and j can be seen as an element of R, we can apply i to j, thus obtaining a new elementary embedding denoted $i[j]$. By construction, the bracket operation on elementary embeddings is left self-distributive, and the main result is that the Laver tables A_n are natural quotients of the LD-systems of elementary embeddings thus obtained. This connection results in a dictionary between elementary embeddings in set theory and values in the Laver tables. In particular, the Laver–Steel theorem, a deep well-foundedness result about elementary embeddings, translates into the result that the number of values occurring in the first row of A_n tends to infinity with n. The puzzling point is that no direct proof of the latter combinatorial statement has been discovered so far. So, as the existence of a self-similar rank is an unprovable axiom, the logical status of this statement remains open.

Part C consists of four chapters. In Chapter X, we introduce the Laver tables, we present Drápal's classification of finite monogenic LD-systems, and we mention other families of LD-systems, such as idempotent and two-sided self-distributive LD-systems.

In Chapter XI, we study LD-monoids, which are LD-systems equipped with an additional compatible associative operation. We solve the natural problem of completing a given LD-system into an LD-monoid, and we give a complete description of free LD-monoids. We also study the LD-monoid of extended braids introduced in Chapter I.

In Chapter XII, we describe the LD-systems of elementary embeddings. Here, we give a short self-contained introduction to the relevant facts needed from set theory, and show how self-distributivity naturally appears. Using a well-foundedness argument—the core of set theory—we establish the Laver–Steel theorem, which is crucial in further applications.

In Chapter XIII, we return to the Laver tables and we give a speculative conclusion to the book. We construct the above mentioned dictionary between elementary embeddings and Laver tables. Building on the unprovable hypothesis postulating the existence of a self-similar rank, we deduce results about the values in A_n, and we show why a direct combinatorial proof of the latter results has to be very complicated.

The methods used here are mostly algebraic and combinatorial in nature. The emphasis is put on words, rather than on the elements of a monoid, a group, or an LD-system they represent. Only such an approach allows us to study fine features which are not visible in the monoid, the group or the LD-system because the relations they involve are not compatible with the congruence defining the considered monoid, group, or LD-system. For instance, the notion of braid word reversing introduced in Chapter II is a refinement of the standard notion of braid word equivalence, and both induce equality of braids. However, using the former relation rather than the latter is crucial for defining the action of braids on non-classical LD-systems and deriving applications such as the braid order. This perhaps explains why such an action had not been considered before.

Exercises appear at the end of most sections, usually with the aim of mentioning in a short way further results (a lot of them correspond to unpublished material).

Historical remarks and proper credits are given in the notes at the end of each chapter. One precision is in order. The current text is deliberately centered on the author's approach to the subject—with the noticeable exception of Chapters IV, X, XII, and XIII, which rely on work by Burckel, Laver, Drápal, and Dougherty. However, we by no means claim that all unattributed results are ours: a number of them is part of the folklore, and there is no doubt that Richard Laver has known a lot of them for many years.

It is a pleasure to express my thanks to P. Ageron, G. Basset, S. Burckel, A. Drápal, A. Kanamori, R. Laver, B. Leclerc, J. Lescot, A. Sossinsky, and B. Wiest for valuable discussions, comments and corrections. I owe special thanks to T. Kepka, who prepared the historical notes at the end of Chapter X, to M. Picantin, who found uncountably many mistakes in the manuscript, and to C. Kassel, who suggested a great number of improvements.

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Patrick Dehornoy

I

Braids vs. Self-Distributive Systems

This chapter gives a quick introduction to our main subject, namely the connection between braids and self-distributive operations. Our presentation is based on the concrete idea of colouring the strands of a braid, and left self-distributivity arises as a natural compatibility condition. At this early stage, some of the constructions may look artificial or strange: it will be one of the aims of the subsequent chapters, in particular in Part B of this book, to explain them and hopefully make all of them natural.

Section 1 contains an elementary introduction to Artin's braid groups. We establish the standard presentation of the braid group, a basis for further algebraic developments. In Section 2, we describe an action of braids on those algebraic systems that involve a self-distributive operation. We review the classical examples of such systems, and mention the braid properties obtained using the action. We also describe a non-classical example involving injections. In Section 3, we show how a self-distributive operation, called braid exponentiation, can be constructed inside Artin's braid group B_∞ by using braid colourings. We present some related self-distributive systems, and we prove that left division associated with braid exponentiation has no cycle, a crucial result for subsequent order properties. Finally, in Section 4, we consider LD-monoids, which are structures involving both a left self-distributive operation and a compatible associative product. LD-monoids naturally appear when colouring braid diagrams. We construct such a structure on a partial topological completion of the braid group B_∞ corresponding to the intuitive idea of strands vanishing at infinity.

II Word Reversing

The aim of this preparatory chapter is triple. Firstly, we wish to give a "modern" proof of the basic algebraic properties of braid groups. The second aim is to introduce a specific combinatorial technique, called word reversing, which is needed in Chapter III to extend the braid action of Chapter I on LD-systems. The third aim is to develop the techinque of word reversing in a general framework, so as to be able to apply it to a certain group G_{LD} in Chapter VIII.

The chapter is organized as follows. In Section 1, we consider monoid presentations of a particular form, of which the standard presentation of B_n is a typical example, and we define word reversing as a possible method for solving the word problem. In every case, word reversing gives a sufficient condition for two words to represent the same element of the monoid. The condition however is not necessary in general: there can exist pairs of words for which the process does not converge, or it does but it gives a wrong answer. In Sections 2 and 3, we give criteria for avoiding such problems. In Section 2, we introduce coherence, which guarantees that word reversing gives a correct answer. We show that, when an additional Noetherianity condition is satisfied, coherence is a consequence of a local assumption which can be checked effectively. When these conditions are satisfied, the considered monoid admits a nice theory of left divisibility, and word reversing is connected with the computation of right lcm's. In Section 3, we consider the termination of word reversing, and, again, we give a sufficient condition, namely the existence of a special element that we call a Garside element. Under such conditions, the monoid embeds in a group of fractions, the latter is torsion free and its word problem is solvable by a double word reversing. In Sections 4 and 5, we show that all technical hypotheses considered in Sections 2 and 3 are satisfied in the case of braid groups, and deduce a number of algebraic properties of the monoids B_n^+ and the groups B_n.

III
The Braid Order

In this chapter, we construct a linear ordering of the braids. This linear ordering appears canonical in several respects; in particular, the braids larger than 1 are characterized as those braids admitting an expression where the generator with smaller index appears positively only, *i.e.*, where some letter σ_i occurs, but the letter σ_i^{-1} does not, nor does any letter $\sigma_k^{\pm 1}$ with $k < i$. The order is decidable, *i.e.*, there exists an effective algorithm that compares any two given braid words, it is compatible with multiplication on one side, and the set B_∞ of all braids is order isomorphic to the rationals.

The organization of the chapter is as follows. In Section 1, we show how to define a partial action of braids on the powers of an arbitrary left cancellative LD-system using the word reversing technique of Chapter II. In Section 2, we construct the braid order. To this end, we introduce special braids as those braids that can be obtained from the unit braid using the exponentiation of Section I.3, and, by using a general property of monogenic LD-systems that will be established in Chapter V, we first construct a linear order on special braids. Then, using the partial action of braids on the LD-system consisting of special braids equipped with braid exponentiation, we extend the linear order on special braids into a linear order $<_L$ on arbitrary braids. In Section 3, we describe a geometrical algorithm that compares braids with respect to $<_L$. This algorithm is very efficient in practice. In Section 4, we give three alternative definitions of $<_L$, one in terms of automorphisms of a free group, one in terms of homeomorphisms of a punctured disk, and, finally, one in terms of an action of braids on the real line connected with hyperbolic geometry.

IV
The Order on Positive Braids

We investigate now the restriction of the linear braid order of Chapter III to positive braids, *i.e.*, to those braids that admit an expression where no letter σ_i^{-1} occurs. The main result is Laver's theorem that the restriction of $<_L$ to B_n^+ is a well ordering. Here we follow Burckel's approach, which associates with every positive braid a normal form consisting in a finite tree; the order of positive braids is then a lexicographical ordering for the associated trees, and one deduces that the order type of $(B_n^+, <_L)$ is the ordinal $\omega^{\omega^{n-2}}$.

As the general construction is intricate, we first consider the special case of 3-strand braids in Section 1, and give a complete proof in this case. Here, things are very simple, as the involved trees can be identified with sequences of integers. In Section 2, we describe the general case, which is similar yet combinatorially more complicated. In Section 3, we give some applications of the properties of the order on positive braids, in particular, we prove that the linear order $<_L$ extends the subword order of positive words, and we present a conjecture of Laver about the action of braids on braids. In an appendix, we give the first elements in the well ordering on B_4^+, together with the associated trees and their ordinal rank.

V
Orders on Free LD-systems

Here, we begin our study of free LD-systems, which will be continued in the following four chapters. The main result of this chapter is that every free LD-system admits a canonical linear ordering. We deduce a solution for the word problem of left self-distributivity, and a simple criterion for recognizing free LD-systems.

The chapter is organized as follows. In Section 1, we recall the general construction of free algebraic systems, and we describe free LD-quasigroups. In Section 2, we introduce the notion of LD-equivalent terms, and we establish a general property of monogenic LD-systems called absorption by right powers. In Section 3, we introduce the notion of being an LD-expansion, a refinement of being LD-equivalent, and we show that two terms are LD-equivalent if and only if they admit a common LD-expansion (confluence property). This result is used in Section 4 to establish the comparison property used in Chapter III when constructing the braid order. In Section 5, we use the iterated left divisibility relation to construct a linear order on every free LD-system. Finally, in Section 6, we deduce that the word problem of the left self-distributivity identity is solvable and we establish Laver's criterion for a given LD-system to be free. This leads to realizations of the free LD-system of rank 1 inside the braid group B_∞, and of the free LD-systems of any rank inside some extension of B_∞.

VI
Normal Forms

In Chapter V, we solved the word problem of LD-equivalence by constructing an effective algorithm directly recognizing whether two given terms are LD-equivalent. Here, we prove normal form results: we construct several families of distinguished terms such that every term is LD-equivalent to exactly one term in the family. This gives alternative solutions to the word problem, as two terms are LD-equivalent if and only if their normal forms are equal, but also new applications.

The main technical notion is that of a cut of a term. For t_0 a fixed term, we define a cut of t_0 to be a term obtained from t_0 by ignoring what lies at the right of some variable. So, there are as many cuts in t_0 as there are occurrences of variables. Then, we define a fractional cut of t_0 to be an iterated product of cuts of t_0 that, in some sense, interpolates between the cuts of t_0. If there are n occurrences of variables in t_0, the cuts of t_0 can be numbered $1, \dots, n$, and, then, the fractional cuts of t_0 can be specified by rational numbers between 1 and n, typically 2.1 or 3.101. The main result of the chapter states that every term t satisfying $t \sqsubseteq_{LD} t_0$ is LD-equivalent to a unique fractional cut of t_0, naturally called the t_0-normal form of t, and, therefore, the LD-class of t can be specified by a rational number as above. The result extends to convenient infinite terms t_0, and, in particular, every term in T_1 admits a well-defined x^{∞}-normal form.

The chapter comprises five sections. In Section 1, we develop a geometrical framework for working with terms viewed as binary trees, and, in particular, for specifying the subterms by using addresses. In Section 2, we introduce the cuts of a term, and establish their basic properties, in particular their behaviour in LD-expansions. In Section 3, we prove a first normal form result. We introduce the notion of a ∂-normal term, and prove that every term is LD-equivalent to a ∂-normal term. The remarkably simple argument relies on the absorption property of Section V.2. In Section 4, we introduce a new family of normal terms connected with the fractional cuts alluded above. The proof that every term is LD-equivalent to a normal term starts from the existence of the ∂-normal form. Section 5 is devoted to applications of the previous results: we prove that the inequality $b \sqsubset ab$ holds for all a, b in the free LD-system FLD_1, we study left division in FLD_1, and we deduce a complete description of special braids.

VII
The Geometry Monoid

Studying free LD-systems amounts to studying LD-equivalence of terms. Now $t =_{LD} t'$ holds if t can be transformed into t' by iteratively applying the left self-distributivity identity, which can be seen as applying some operator that specifies where and how (from left to right or from right to left) the identity is applied. Applying the identity several times amounts to composing the associated operators. So we obtain a monoid of (partial) operators acting on terms, so that the LD-equivalence class of a term is its orbit under the action. The aim of this chapter is to study the monoid $\mathcal{G}_{LD}$ involved in this action, which we call the geometry monoid of (LD) as it captures a number of geometrical relations involving left self-distributivity.

The chapter is organized as follows. In Section 1, we introduce the operators LD_w and the monoid $\mathcal{G}_{LD}$ they generate. We describe the domain and image of LD_w, and we interpret the product in $\mathcal{G}_{LD}$ as a term unification process. In Section 2, we use geometric arguments to build a list of relations, called LD-relations, that hold in the monoid $\mathcal{G}_{LD}$. We do not prove that LD-relations form a presentation of $\mathcal{G}_{LD}$, but we show in Section 3 and 4 how to use them to re-prove a number of previously known properties of left self-distributivity. In particular, we give a syntactic proof of the confluence property, *i.e.*, one that resorts to LD-relations only. To this end, we introduce for each term t a distinguished word $\boldsymbol{\Delta}_t$ which describes the passage from t to ∂t, and which is directly reminiscent of Garside's braid word $\boldsymbol{\Delta}_n$..

VIII
The Group of Left Self-Distributivity

The geometry monoid $\mathcal{G}_{LD}$ that describes the action of the left self-distributivity identity on terms is not a group, and, contrary to the case of associativity, we cannot obtain a group by merely using a projection. However, we know a family of relations holding in $\mathcal{G}_{LD}$, namely these LD-relations that define $\equiv^+$, and we have observed that many properties of $\mathcal{G}_{LD}$ can be established by using these relations exclusively. In this chapter, we investigate the abstract group G_{LD} for which LD-relations form a presentation. The hypothesis that G_{LD} must resemble $\mathcal{G}_{LD}$ is kept as a leading principle, and indeed we can show that all geometrical parameters defined in $\mathcal{G}_{LD}$ admit counterparts in G_{LD}. On the other hand, the group G_{LD} turns out to be an extension of the braid group B_∞—this being the precise content of our slogan: "The geometry of left self-distributivity is an extension of the geometry of braids." Many results about G_{LD} and B_∞ originate in this connection. In particular, braid exponentiation and braid ordering come from an operation and a relation on G_{LD} that somehow explain them and make their construction natural.

Technically, the group G_{LD} behaves like a sort of generalized Artin group. It shares several properties with such groups, yet a number of technical problems arise from G_{LD}, contrary to B_∞, not being a direct limit of finite type groups.

The divisions of the chapter are as follows. In Section 1, we introduce the group G_{LD} and the corresponding monoid M_{LD}. We observe that the braid group B_∞ is a quotient of G_{LD}, a result connected with the action of braids on left self-distributive systems. We also observe that the presentation of G_{LD} is associated with a complement, and verify that this complement satisfies all conditions of Chapter II. In Section 2, we embed T_1 into G_{LD} by using the absorption property of Chapter V to associate with every term a distinguished word that describes its construction. We deduce a complete description of the connection between $\mathcal{G}_{LD}$ and G_{LD}, and explain how braid exponentiation arises. By extending the approach to the case of several variables, we show how charged braids then appear naturally. In Section 3, we introduce two different (pre)-orders on the group G_{LD}. The first is a preorder connected with the braid ordering, and using it gives a purely syntactical proof for the acyclicity property of free LD-systems, one that does not use braid exponentiation. The second relation is a linear ordering on G_{LD} which is compatible with multiplication on both sides. In Section 4, we show that shifting the addresses gives a family of injective endomorphisms of G_{LD}, which amounts to determining certain parabolic subgroups of G_{LD}. Finally, we introduce in Section 5 the notion of a simple element in the monoid M_{LD}, which is an exact analog of the notion of a simple braid in B_∞^+. We establish in particular a normal form result in M_{LD} which directly extends the braid normal form of Chapter II.

IX
Progressive Expansions

In this chapter, we deepen our study of left self-distributivity and, describing the geometry of the terms $\partial^k t$ more closely, we prove partial results about the convergence of the Polish Algorithm and the Embedding Conjecture. The Polish Algorithm is a natural syntactic method for deciding LD-equivalence of terms, and the question of whether it always converges is one of the most puzzling open questions about left self-distributivity. The Embedding Conjecture claims that the monoid M_{LD} embeds in the group G_{LD}. The main technical notion in the chapter is the notion of a progressive LD-expansion, a particular kind of LD-expansion where self-distributivity is applied to positions that move from from left to right only. Its interest lies in the uniqueness properties it entails.

The organization of the chapter is as follows. In Section 1, we introduce the Polish Algorithm and the notion of a progressive word. In Section 2, we define the notion of an element of FLD_∞ *appearing* in a term, and we study an associated covering relation. In Section 3, we continue the analysis of the terms $\partial^k t$ initiated in Chapter VI. We prove that the latter terms satisfy a strong self-similarity property called perfectness, and we deduce partial results about the practical determination of the t_0-normal form of a term. In Section 4, we deduce convergence results for the Polish Algorithm; we establish in particular that the algorithm always converges when running on terms that are $\sqsubset_{LD}$-comparable to injective terms. We also extend the results established in Section VIII.5 for simple elements of M_{LD} to a more general notion of degree k elements where $\partial^k t$ replaces ∂t. In Section 5, we establish explicit decompositions for the elements Δ_t, which we recall are the counterparts to Garside's braids Δ_n. In Section 6, we prove a number of particular instances of the Embedding Conjecture by establishing the existence of certain sets of terms called confluent families. Finally, we conclude this chapter and Part B with a brief appendix describing what remains from the current approach when the left self-distributivity identity is replaced with another algebraic identity, in particular associativity.

X
More LD-Systems

So far we have described several examples of LD-systems: conjugacy of a group, braid exponentiation, injection bracket. In this chapter, we give new examples, and mention some general results about LD-systems, and about those special families obtained by prescribing additional identities.

The organization of the chapter is as follows. In Section 1, we introduce an infinite family of finite monogenic LD-systems that we call the Laver tables. For each n, there exists one Laver table with 2^n elements, and these tables organize into a projective system. Here we establish their basic properties. In Section 2, we sketch Drápal's result that every finite monogenic LD-system can be constructed from a Laver table using some uniform scheme. In Section 3, we introduce multi-LD-systems, which are sets equipped with several mutually left distributive operations, and we describe free multi-LD-systems in terms of free LD-systems. In Section 4, we consider idempotents in LD-systems, and LDI-systems, which are those LD-systems where every element is idempotent. We establish Joyce's result that group conjugacy is axiomatized by the axioms of LD-quasigroups plus idempotency when both "aba^{-1}" and "$a^{-1}ba$" are considered, and Larue's result that group conjugacy is not axiomatized by (LD) plus idempotency when "aba^{-1}" only is used. Finally, in Section 5, we consider LRD-systems, which are those LD-systems that also satisfy the right self-distributivity law. The main result relates divisible LRD-systems with commutative Moufang loops. In an appendix, we explicitly display the Laver tables A_n for $n \leq 6$, and, partially, for $n \leq 10$.

XI
LD-Monoids

A number of LD-systems can be equipped with a second, associative operation connected to the self-distributive operation by several mixed identities. Here we study such double structures, called LD-monoids. In particular, we discuss the question of completing a given LD-system into an LD-monoid. We give two solutions, and deduce a complete description of free LD-monoids. The global conclusion is that the self-distributive operation is the core of the structure, and that adding an associative product is essentially trivial. However, the case of braid exponentiation is not so simple, and applying the above mentioned completion scheme requires considering the extended braids of Section I.4.

The chapter is organized as follows. In Section 1, we give examples of LD-monoids, and discuss the problem of completing a given monoid into an LD-monoid. In Section 2, we address the problem of embedding a given LD-system into an LD-monoid; we describe a universal solution called the free completion, as well as another completion defined using composition of left translations. The construction applies in particular to the Laver tables. In Section 3, we show that the free LD-system FLD_X embeds in the free LD-monoid based on X, and that the latter can be constructed inside FLD_X. This enables us in particular to solve the word problem of free LD-monoids. In Section 4, we deduce that every free LD-monoid admits canonical linear orderings. As for free LD-systems, the existence of such orderings leads to a freeness criterion and to various algebraic properties like left cancellativity. Finally, in Section 5, we come back to the LD-monoid of extended braids as defined in Chapter I, and show that it includes many free LD-monoids of rank 1.

XII
Elementary Embeddings

Here, we describe some left self-distributive structures that appear in set theory when mappings called elementary embeddings are iterated. We try to give a self-contained exposition of the subject by extracting those minimal results necessary for the construction, and assuming no knowledge of set theory. However, the chapter can be used as a black box leading to exportable results like Proposition 4.11. Actually, it can even be skipped, for it only consists in one more example—but one that has played a crucial role in the subject, and still does, as some of the algebraic results it leads to have so far received no alternative proof, as we shall see in Chapter XIII.

The organization of the chapter is as follows. In Section 1, we introduce large cardinals in set theory. In Section 2, we define the notions of an elementary embedding and of a self-similar rank, and we explain why the existence of such objects cannot be proved in ordinary mathematics. We associate with every nontrivial elementary embedding a distinguished ordinal called its critical ordinal. In Section 3, we show how a left self-distributive operation arises on elementary embeddings associated with a self-similar rank. The main result here is the Laver–Steel theorem, a deep consequence of the fact that ordinals are well ordered: it asserts that the critical ordinals of a certain type of sequence of elementary embeddings have no upper bound. Using this result, one shows that left division in the LD-system constructed above has no cycle. Finally, we study in Section 4 the finite quotients of this LD-system, and the Laver tables of Chapter X appear naturally.

XIII
More about the Laver Tables

In this chapter, we come back to the finite Laver tables introduced in Chapter X. We show how using the results of set theory established in Chapter XII gives new properties of A_n, in particular the result that the period in the first row of A_n goes to infinity with n. The unusual point here is that the result is established using the existence of a self-similar rank as an hypothesis, and we know that this is an unprovable logical statement. Thus the previous argument is not a proof in the usual sense, and another proof has to be found, typically a combinatorial argument involving only intrinsically finite objects. Unfortunately (or fortunately...), no such proof is known to date, and it is only known that such a proof, if it exists, has to be very complicated in some sense.

The chapter is organized as follows. In Section 1, we construct a dictionary between the iterations of an elementary embedding as described in Chapter XII and the finite tables A_n of Section X.1, and we use it to translate the Laver–Steel theorem into results about the systems A_n. In Section 2, we sketch partial combinatorial results toward those properties established in Section 1 using elementary embeddings. In Section 3, we come back to elementary embeddings, and we show that computing some critical ordinals requires using fast growing functions on the integers. Finally, in Section 4, we deduce from the previous results a huge lower bound for the least n such that the period in the first row of A_n goes beyond 16. The existence of such bounds suggests that no simple combinatorial proof of the results of Section 1 is likely to exist, which is confirmed by the result that no such proof can be formalized inside the logical system called Primitive Recursive Arithmetic.