ON REPRESENTATIONS OF BRAIDS AS AUTOMORPHISMS OF FREE GROUPS AND CORRESPONDING LINEAR REPRESENTATIONS

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Abstract. In this survey we explore relationships between several different representations of braid groups as automorphisms of free groups as well as induced linear representations.

1. Introduction

In his first seminal paper on braid groups [1], Artin proposed an interpretation of the braid group B_n as a group of automorphisms of the free group F_n. This representation has several important properties: for instance it gives an immediate solution for the word problem, and, using Fox derivatives, one can construct Burau representation. Actually, the relevance of Artin representation in the study of braids, mapping class groups and knots is impressive and it motivated to look forward for generalizations or other "geometric" representations (see for instance [2, 5, 13–15]).

There are several other faithful representations of braids in terms of automorphisms of free groups: in the following we will recall in particular Perron-Vannier representation [13], Wada representations [15] and we will propose a new representation, that we will call Fenn-Rolfsen-Zhu representation because inspired from [7].

We refer to [2] for a complete survey on braids seen as automorphisms of free groups and for algebraical proofs of well known results arising from this approach: the main aim of this note is to construct some reductions and extensions of above mentioned representations and to provide several algebraical relations between them. In particular we will show that they are all faithful, that extended Artin representation is conjugated to Fenn-Rolfsen-Zhu representation (Theorem 2.3), that reduced Fenn-Rolfsen-Zhu representation is conjugated to Artin representation (Proposition 2.5) and that extended Perron-Vannier representation is actually a Wada representation (Theorem 4.1). At the end of Section 2 we will construct a family of representations of B_n containing reduced Artin representation, extended Artin representation and Fenn-Rolfsen-Zhu representation. In Section 5 we will

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provide linear representation of B_n induced by Perron-Vannier and Fenn-Rolfsen-Zhu representations (see in particular Proposition 5.4): as a corollary we will show that extended Perron Vannier representation is not equivalent to extended Artin representation (Proposition 5.3).

2. Representations of Artin and Fenn-Rolfsen-Zhu

Let F_m be the free group of rank n with the set of free generators $\{x_1, x_2, \ldots, x_m\}$. Assume also that $\mathrm{Aut}(F_m)$ is the automorphism group of F_m.

In the following we will show extensions and reductions of several representations of B_n into $\mathrm{Aut}(F_m)$ (for some m) and we will establish relations between them; in particular we will remark when they are *conjugated*.

Definition 2.1. *Let $n, m > 1$. Two representations $\rho, \rho' : B_n \to Aut(F_m)$ are conjugated if there exists an automorphism $\chi : F_m \to F_m$ such that $\chi \circ \rho(\beta) \circ \chi^{-1} = \rho'(\beta)$ for all $\beta \in B_n$.*

We will consider also a weaker notion of equivalence for representations (see Definition 1.4 of [5]).

Definition 2.2. *Let $n, m > 1$. Two representations $\rho, \rho' : B_n \to Aut(F_m)$ are equivalent if there exist automorphisms $\chi : F_m \to F_m$ and $\mu : B_n \to B_n$ such that $\chi \circ \rho(\beta) \circ \chi^{-1} = \rho'(\mu(\beta))$ for all $\beta \in B_n$.*

When $n = m$ the first famous example of representation is the Artin representation of B_n. This representation

$$\rho_A : B_n \longrightarrow \mathrm{Aut}(F_n),$$

due to Artin himself, is defined associating to any generator σ_i, for $i = 1, 2, \ldots, n-1$, of B_n the following automorphism of F_n:

$$\rho_A(\sigma_i) : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

Let us recall also that there is a geometrical interpretation of ρ_A: the braid group B_n is isomorphic to the mapping class group of the n-punctured disk, that we denote by $\mathbb{D}_n$, and Artin representation therefore corresponds to the induced action of B_n on $\pi_1(\mathbb{D}_n) = F_n$.

We will consider an extension of Artin representation: $\widetilde{\rho}_A : B_n \longrightarrow \mathrm{Aut}(F_{n+1})$, where $F_{n+1} = \langle x_0, x_1, \ldots, x_n \rangle$, defining

$$\widetilde{\rho}_A(\sigma_i) : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

for all generators $x_0, x_1, x_2, \ldots, x_n$.

In [7], Fenn, Rolfsen and Zhu constructed an action of B_n on particular arcs of the n-punctured disk $\mathbb{D}_n$. This construction can be described as follows.

Let $\mathcal{P} = p_1, \ldots, p_n$ be the set of punctures of $\mathbb{D}_n$ and let A be an oriented arc with endpoints in $\mathcal{P}$. To A we can associate a word in the symbols $I_0, I_1, \ldots, I_n$, $I_0^{-1}, I_1^{-1}, \ldots, I_n^{-1}$. Let suppose that the punctures $p_1, \ldots, p_n$ on $\mathbb{D}_n$ are on the real line and let $s_0, \ldots, s_{n+1}$ be the segments on Figure 1. Assume that A is transverse to the real line: starting from the initial point of A, say p_k, and write I_m when

A crosses the segment s_m with increasing imaginary part and write I_m^{-1} otherwise. Since B_n acts on (isotopy classes of) arcs, B_n acts on the word $w(A)$ associated to A.[1]

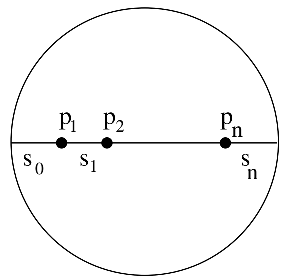

FIGURE 1. The group B_n acts on the segments $s_0, \dots, s_{n+1}$.

In particular the generator σ_i acts on any letter of $w(A)$ (suppose that A has different endpoints than i and $i+1$) as follows:

$$\begin{array}{ll} I_i \longmapsto I_{i-1}\, I_i^{-1}\, I_{i+1}, & \\ I_i^{-1} \longmapsto I_{i+1}^{-1}\, I_i\, I_{i-1}^{-1}, & \\ I_l^{\pm 1} \longmapsto I_l^{\pm 1}, & l \neq i. \end{array}$$

This construction inspired us the following representation:

$$\rho_F : B_n \longrightarrow \mathrm{Aut}(F_{n+1}),\ F_{n+1} = \langle y_0, y_1, \dots, y_n \rangle,$$

which maps the generators of B_n to the following automorphisms

$$\rho_F(\sigma_i) : \begin{cases} y_i \longmapsto y_{i-1}\, y_i^{-1}\, y_{i+1}, & \\ y_l \longmapsto y_l, & l \neq i. \end{cases}$$

Remark that previous representation is evidently equivalent (see also the end of the section) to the reduced Artin representation (see [4, p. 121]) $\rho_{RA} : B_n \longrightarrow \mathrm{Aut}(F_{n+1})$:

$$\rho_{RA}(\sigma_i) : \begin{cases} y_i \longmapsto y_{i+1} y_i^{-1} y_{i-1}, & \\ y_j \longmapsto y_j, j \neq i, & \end{cases} \qquad i = 1, 2, \dots, n-1.$$

One can easily check that $\rho_{RA}(\sigma_i) = \rho_F(\sigma_i^{-1})$. The relation between representations ρ_A and ρ_F can be algebraically described as follows.

Theorem 2.3. *The representation $\rho_F : B_n \longrightarrow Aut(F_{n+1})$ is conjugated to the representation $\widetilde{\rho}_A : B_n \longrightarrow Aut(F_{n+1})$. In particular, ρ_F is faithful.*

Proof. We define the elements $y_0 = x_0, y_1 = x_1^{-1} y_0 = x_1^{-1} x_0,\ y_2 = x_2^{-1} y_1 = x_2^{-1} x_1^{-1} x_0, \dots,\ y_n = x_n^{-1} y_{n-1} = x_n^{-1} x_{n-1}^{-1} \dots x_1^{-1} x_0$ in $F_{n+1} = \langle x_0, x_1, \dots, x_n \rangle$. It is evident that these elements is a basis of F_{n+1} and the old basis can be express from new by the rules

$$x_0 = y_0,\ x_1 = y_0 y_1^{-1},\ x_2 = y_1 y_2^{-1}, \dots, x_n = y_{n-1} y_n^{-1}.$$

[1]We did not show that $w(A)$ is invariant up to isotopy. In [7] it is explained how to associate an unique word $w(A)$ to a given arc A.

Let us find the Artin representation in the new basis $\{y_0, y_1, \ldots, y_n\}$. We have [2]

$$
\begin{aligned}
y_k^{\widetilde{\rho}_A(\sigma_i)} &= y_k, \ k < i, \\
y_i^{\widetilde{\rho}_A(\sigma_i)} = (x_i^{-1} y_{i-1})^{\widetilde{\rho}_A(\sigma_i)} &= x_i x_{i+1}^{-1} x_i^{-1} y_{i-1} = y_{i-1} y_i^{-1} y_{i+1}, \\
y_{i+1}^{\widetilde{\rho}_A(\sigma_i)} = (x_{i+1}^{-1} y_i)^{\widetilde{\rho}_A(\sigma_i)} &= x_i^{-1} y_{i-1} y_i^{-1} y_{i+1} = y_{i+1}, \\
y_l^{\widetilde{\rho}_A(\sigma_i)} = (x_l^{-1} y_{l-1})^{\widetilde{\rho}_A(\sigma_i)} &= y_l, \ l > i+1.
\end{aligned}
$$

Hence the Artin representation in the bases $\{y_0, y_1, \ldots, y_n\}$ coincides with the representation of Fenn-Rolfsen-Zhu. □

We can reformulate previous theorem saying that the representations $\widetilde{\rho}_A$ and ρ_F are conjugated (see Definition 2.1).

Let $\varphi_{AF} \in \mathrm{Aut}(F_{n+1})$ be the automorphism such that

$$\varphi_{AF}^{-1}\, \widetilde{\rho}_A\, \varphi_{AF} = \rho_F,$$

or, in other words, such that for any generator x_i of F_{n+1} and for any $\sigma_j \in B_n$ we get

$$x_i^{\varphi_{AF}^{-1}\, \widetilde{\rho}_A(\sigma_j)\, \varphi_{AF}} = x_i^{\rho_F(\sigma_j)}, \ i = 0, 1, \ldots, n, \ j = 1, 2, \ldots, n-1.$$

We can also determine φ_{AF}:

$$
\varphi_{AF} : \begin{cases}
x_0 \longmapsto x_0, \\
x_1 \longmapsto x_0 x_1^{-1}, \\
x_2 \longmapsto x_1 x_2^{-1}, \\
\vdots \\
x_n \longmapsto x_{n-1} x_n^{-1}.
\end{cases}
$$

Then

$$
\varphi_{AF}^{-1} : \begin{cases}
x_0 \longmapsto x_0, \\
x_1 \longmapsto x_1^{-1} x_0, \\
x_2 \longmapsto x_2^{-1} x_1^{-1} x_0, \\
\vdots \\
x_n \longmapsto x_n^{-1} x_{n-1}^{-1} \ldots x_1^{-1} x_0,
\end{cases}
$$

and we can check the formulas

$$x_i^{\varphi_{AF}^{-1}\, \widetilde{\rho}_A(\sigma_j)\, \varphi_{AF}} = x_i^{\rho_F(\sigma_j)}, \ i = 0, 1, \ldots, n, \ j = 1, 2, \ldots, n-1.$$

We know that every automorphism in $\rho_A(B_n)$ fixes the product $x_1 x_2 \ldots x_n$. Hence every automorphism in $\widetilde{\rho}_A(B_n)$ fixes the product $x_0^k x_1 x_2 \ldots x_n$ for arbitrary integer k. For the automorphisms in $\rho_F(B_n)$ we have a similar result.

Corollary 2.4. *Any automorphism in $\rho_F(B_n)$ fixes elements w in the subgroup $\langle x_0, x_n \rangle$.*

Proof. It is enough to show that elements $w = (x_0^k x_1 x_2 \ldots x_n)^{\varphi_{AF}} = x_0^{k+1} x_n^{-1}$, where $k \in \mathbb{Z}$, are fixed by $\rho_F(B_n)$. We know that any element $x_0^k x_1 x_2 \ldots x_n$ is fixed by every automorphism in $\widetilde{\rho}_A(B_n)$. Hence, if we define w by the formula $w = (x_0^k x_1 x_2 \ldots x_n)^{\varphi_{AF}}$ then

$$w^{\varphi_{AF}^{-1}\, \widetilde{\rho}_A(B_n)\, \varphi_{AF}} = (x_0^k x_1 x_2 \ldots x_n)^{\widetilde{\rho}_A(B_n)\, \varphi_{AF}} = (x_0^k x_1 x_2 \ldots x_n)^{\varphi_{AF}} = w.$$

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[2] In the following, given $\rho : B_n \to \mathrm{Aut}(F_m)$, we will note by $x^{\rho(y)}$ the action on $x \in F_m$ by $\rho(y)$.

Since,

$$\varphi_{AF}^{-1}\,\widetilde{\rho}_A(B_n)\,\varphi_{AF} = \rho_F(B_n),$$

then

$$w^{\rho_F(B_n)} = w.$$

□

We can define a representation $\rho_{RF} : B_n \longrightarrow \mathrm{Aut}(F_n)$ that is the composition of ρ_F and the homomorphism which forgets y_0, i.e.

$$\rho_{RF}(\sigma_1) : \begin{cases} y_1 \longmapsto y_1^{-1}\, y_2, & \\ y_j \longmapsto y_j, & j \neq 1, \end{cases}$$

$$\rho_{RF}(\sigma_i) : \begin{cases} y_i \longmapsto y_{i-1}\, y_i^{-1}\, y_{i+1}, & \\ y_j \longmapsto y_j, & j \neq 1, \end{cases} \quad 1 < i \leq n-1.$$

We can provide a result similar to Theorem 2.3, relating Artin representation ρ_A to representation ρ_{RF}.

Proposition 2.5. *The representation ρ_{RF} is conjugated to the Artin representation ρ_A. In particular, the representation ρ_{RF} is faithful.*

Proof. Take the new generators of $F_n = \langle x_1, x_2, \ldots, x_n \rangle$:

$$y_1 = x_1^{-1},\ y_2 = x_2^{-1} y_1 = x_2^{-1} x_1^{-1}, \ldots, y_n = x_n^{-1} y_{n-1} = x_n^{-1} \ldots x_1^{-1}.$$

Express the old generators

$$x_1 = y_1^{-1},\ x_2 = y_1 y_2^{-1}, \ldots, x_n = y_{n-1} y_n^{-1}.$$

Then the representation ρ_A in the new generators has the form

$$\rho_A(\sigma_1) : \begin{cases} y_1 \longmapsto y_1^{-1}\, y_2, & \\ y_j \longmapsto y_j, & j \neq 1, \end{cases}$$

$$\rho_A(\sigma_i) : \begin{cases} y_i \longmapsto y_{i-1}\, y_i^{-1}\, y_{i+1}, & \\ y_j \longmapsto y_j, & j \neq 1, \end{cases} \quad 1 < i \leq n-1.$$

Therefore we recover representation ρ_{RF}. □

Generalizing previous results, we can construct a larger family of faithful representations that contains ρ_F and ρ_{RA}. For this purpose define the family of automorphisms

$$\varphi_{\varepsilon,\mu,k} : \begin{cases} x_0 \longmapsto x_0^{\varepsilon_0}, \\ x_1 \longmapsto (x_0^{k_1} x_1^{\varepsilon_1})^{\mu_1}, \\ x_2 \longmapsto (x_1^{k_2} x_2^{\varepsilon_2})^{\mu_2}, \\ \vdots \\ x_n \longmapsto (x_{n-1}^{k_n} x_n^{\varepsilon_n})^{\mu_n}, \end{cases}$$

where

$$\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n),\ \varepsilon_i \in \{\pm 1\},\ \mu = (\mu_1, \mu_2, \ldots, \mu_n),\ \mu_j \in \{\pm 1\},$$

$$k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n.$$

Then the inverse automorphism to $\varphi_{\varepsilon,\mu,k}$ is equal to

$$\varphi_{\varepsilon,\mu,k}^{-1} : \begin{cases} x_0 \longmapsto x_0^{\varepsilon_0}, \\ x_1 \longmapsto (x_0^{-\varepsilon_0 k_1} x_1^{\mu_1})^{\varepsilon_1}, \\ x_2 \longmapsto ((x_0^{-\varepsilon_0 k_1} x_1^{\mu_1})^{-\varepsilon_1 k_2} x_2^{\mu_2})^{\varepsilon_2}, \\ \vdots \\ x_n \longmapsto ((\ldots((x_0^{-\varepsilon_0 k_1} x_1^{\mu_1})^{-\varepsilon_1 k_2} x_2^{\mu_2})^{\varepsilon_2} \ldots x_{n-1}^{\mu_{n-1}})^{-\varepsilon_{n-1} k_n} x_n^{\mu_n})^{\varepsilon_n}, \end{cases}$$

and we can define a representation of B_n by the rule

$$\varphi_{\varepsilon,\mu,k}^{-1}\, \widetilde{\rho}_A(B_n)\, \varphi_{\varepsilon,\mu,k}.$$

In particular, if we take

$$\varepsilon = (1, -1, -1, \ldots, -1) \in \mathbb{Z}^{n+1},\ \mu = (1, 1, \ldots, 1) \in \mathbb{Z}^n,\ k = (1, 1, \ldots, 1) \in \mathbb{Z}^n,$$

then we can define

$$\varphi_{\varepsilon,\mu,k}^{-1}\, \widetilde{\rho}_A(B_n)\, \varphi_{\varepsilon,\mu,k}$$

which is exactly the representation $\rho_F(B_n)$; if we take

$$\varepsilon = (1, 1, \ldots, 1) \in \mathbb{Z}^{n+1},\ \mu = (1, 1, \ldots, 1) \in \mathbb{Z}^n,\ k = (-1, -1, \ldots, -1) \in \mathbb{Z}^n,$$

then

$$\varphi_{\varepsilon,\mu,k}^{-1}\, \widetilde{\rho}_A(B_n)\, \varphi_{\varepsilon,\mu,k}$$

is the reduced Artin representation ρ_{RA}.

3. Perron-Vannier representation

Another interesting faithful representation of the braids as automorphisms of free groups is the Perron-Vannier representation [13].

This representation becomes from the mapping which sends B_n into the mapping class group of the surface Σ shown in Figure 2, where any generator σ_i of B_n is sent into the Dehn twist τ_i along the curve c_i. Perron-Vannier representation is therefore given by the induced action of τ_i $(i = 1, \ldots, n-1)$ on $\pi_1(\Sigma) = F_{n-1}$ (see [6] for a detailed description of this action and for the geometrical interpretation of Σ as a branched 2-fold cover of $\mathbb{C}$).

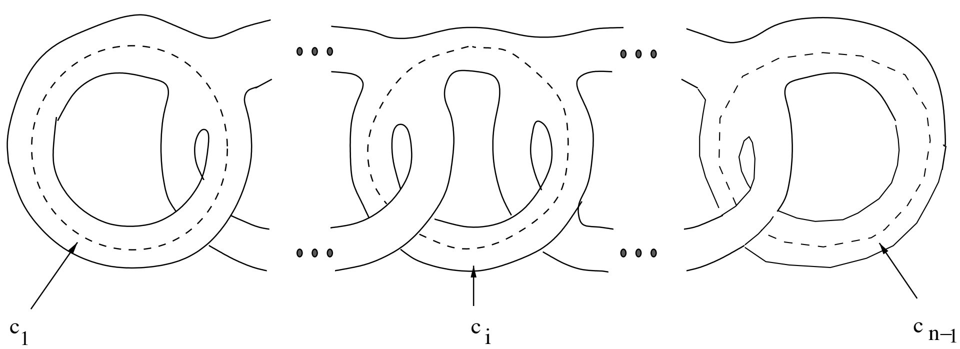

Figure 2. The generator σ_i of B_n is sent into the Dehn twist τ_i along the curve c_i.

Perron-Vannier representation $\rho_P : B_n \to \mathrm{Aut}(F_{n-1})$ is algebraically defined as follows:

$$\rho_P(\sigma_1) : \begin{cases} x_1 \to x_1, \\ x_j \to x_1^{-1} x_j, j \neq 1, \end{cases}$$

and for $2 \leq i \leq n-1$,

$$\rho_P(\sigma_i) : \begin{cases} x_{i-1} \longmapsto x_i, \\ x_i \longmapsto x_i x_{i-1}^{-1} x_i, \\ x_j \longmapsto x_j, & j \neq i-1, i. \end{cases}$$

The faithfulness of ρ_P was proven in [13] with topological arguments (see [6] for an algebraical proof). Starting from Perron-Vannier representation in [6] was constructed another faithful representation $\rho_{CP} : B_n \to \mathrm{Aut}(F_{n-1})$ given algebraically by:

$$\rho_{CP}(\sigma_i) : \begin{cases} y_{i-1} \longmapsto y_{i-1} y_i, \\ y_{i+1} \longmapsto y_i^{-1} y_{i+1} \\ y_j \longmapsto y_j, & j \neq i-1, i+1, \end{cases}$$

where $F_{n-1} = \langle y_1, \ldots, y_{n-1} \rangle$.

In particular, according to the previous definition, we have that:

$$\rho_{CP}(\sigma_1) : \begin{cases} y_1 \longmapsto y_1, \\ y_2 \longmapsto y_1^{-1} y_2 \\ y_j \longmapsto y_j, & j > 2. \end{cases}$$

and

$$\rho_{CP}(\sigma_{n-1}) : \begin{cases} y_j \longmapsto y_j, & j < n-2, \\ y_{n-2} \longmapsto y_{n-2} y_{n-1}, \\ y_{n-1} \longmapsto y_{n-1} \end{cases}$$

We are interested to construct two extensions of Perron-Vannier representations in $\mathrm{Aut}(F_n)$ and $\mathrm{Aut}(F_{n+1})$.

Proposition 3.1. *The following representation $\rho_{CP}^{(1)} : B_n \to Aut(F_n)$ is faithful:*

$$\rho_{CP}^{(1)}(\sigma_1) : \begin{cases} y_1 \longmapsto y_1, \\ y_2 \longmapsto y_1^{-1} y_2 \\ y_j \longmapsto y_j, & j > 2. \end{cases}$$

and for $i > 1$:

$$\rho_{CP}^{(1)}(\sigma_i) : \begin{cases} y_{i-1} \longmapsto y_{i-1} y_i, \\ y_{i+1} \longmapsto y_i^{-1} y_{i+1} \\ y_j \longmapsto y_j, & j \neq i-1, i+1. \end{cases}$$

where $F_n = \langle y_1, \ldots, y_{n-1}, y_n \rangle$.

Proof. The map $F_n \to F_{n-1}$ which "forgets" y_n induces a projection of $q : \rho_{CP}^{(1)}(B_n) \to \rho_{CP}(B_n)$ such that $q \circ \rho_{CP}^{(1)} = \rho_{CP}$. Since $\rho_{CP} : B_n \to \mathrm{Aut}(F_{n-1})$ is faithful, also $\rho_{CP}^{(1)}$ is injective. □

Proposition 3.2. *The following representation $\rho_{CP}^{(2)} : B_n \to Aut(F_{n+1})$ is faithful:*

$$\rho_{CP}^{(2)}(\sigma_i) : \begin{cases} y_{i-1} \longmapsto y_{i-1}y_i, & \\ y_{i+1} \longmapsto y_i^{-1}y_{i+1} & \\ y_j \longmapsto y_j, & j \neq i-1, i+1. \end{cases}$$

where $F_{n+1} = \langle y_0, y_1, \ldots, y_{n-1}, y_n \rangle$.

Proof. We consider the map $F_{n+1} \to F_{n-1}$ which "forgets" y_0 and y_n and we proceed as in previous proposition □

A straightforward consequence of the definition of $\rho_{CP}^{(2)}$ is the following result.

Corollary 3.3. *The images of braids via $\rho_{CP}^{(2)}$ preserve the product $y_0 \cdots y_n$.*

In Proposition 5.3 we will show that ρ_F and extended Perron-Vannier representation $\rho_{CP}^{(2)}$ are not equivalent.

Question [12, Question 17.14]. Let $n \geq 4$. Find the minimal number m for which there is a faithful representation $B_n \longrightarrow \mathrm{Aut}(F_m)$. In particular, is it true that there is no a faithful representation $B_4 \longrightarrow \mathrm{Aut}(F_2)$?

A (non faithful) representation $B_4 \longrightarrow \mathrm{Aut}(F_2)$ was given in [11]: this representation was extended in [10] to a (non faithful) representation $B_{2g} \longrightarrow \mathrm{Aut}(F_{2g-2})$ arising from the action of B_{2g} on a particular ramified double covering.

Remark 3.4. *Inspired by the work in [11], Godelle proposed a representation $\rho_G : B_n \longrightarrow Aut(F_{n-1})$, based on the notion of transvection automorphisms [8]. The representation ρ_G is defined as follows:*

$$\rho_G(\sigma_i) : \begin{cases} x_{i-1} \longmapsto x_i x_{i-1}, & \\ x_{i+1} \longmapsto x_i x_{i+1}, & \\ x_j \longmapsto x_j, & j \neq i \pm 1. \end{cases}$$

for $i \equiv 1\,(mod\,4)$;

$$\rho_G(\sigma_l) : \begin{cases} x_{l-1} \longmapsto x_l^{-1} x_{l-1}, & \\ x_{l+1} \longmapsto x_l^{-1} x_{l+1}, & \\ x_j \longmapsto x_j, & j \neq l \pm 1. \end{cases}$$

for $l \equiv 2\,(mod\,4)$;

$$\rho_G(\sigma_p) : \begin{cases} x_{p-1} \longmapsto x_{p-1} x_p, & \\ x_{p+1} \longmapsto x_{p+1} x_p, & \\ x_j \longmapsto x_j, & j \neq p \pm 1. \end{cases}$$

for $p \equiv 3\,(mod\,4)$;

$$\rho_G(\sigma_q) : \begin{cases} x_{q-1} \longmapsto x_{q-1} x_q^{-1}, & \\ x_{q+1} \longmapsto x_{q+1} x_q^{-1}, & \\ x_j \longmapsto x_j, & j \neq q \pm 1. \end{cases}$$

for $q \equiv 0\,(mod\,4)$,
and where $F_{n-1} = \langle x_1, \ldots, x_{n-1} \rangle$. If we change the basis of F_{n-1} replacing x_i by $y_i = x_i^{-1}$ for $i \equiv 1\,(mod\,4)$, x_l by $y_l = x_l$ for $l \equiv 2\,(mod\,4)$, x_p by $y_p = x_p$ for $p \equiv 3\,(mod\,4)$ and x_q by $y_q = x_q^{-1}$ for $q \equiv 0\,(mod\,4)$ we obtain the Perron-Vannier representation ρ_{CP}.

4. Local type representations

In [15] Wada introduced a family of representations of B_n in $\mathrm{Aut}(F_n)$ of the following special form: any generator σ_i of B_n acts trivially on generators of F_n except a pair of generators:

$$\begin{aligned} x_i^{\sigma_i} &= u(x_i, x_{i+1})\,, \\ x_{i+1}^{\sigma_i} &= v(x_i, x_{i+1})\,, \\ x_j^{\sigma_i} &= x_j \qquad j \neq i, i+1\,, \end{aligned}$$

where u and v are now words in the generators x_i, x_{i+1}, with $\langle x_i, x_{i+1}\rangle \simeq F_2$. Wada named them as shift type representations, but they are usually known as representation of local type.

Wada found seven families of representations of local type (we denote by ψ_j the corresponding representation):

- Type 1, ψ_1: $u(x_i, x_{i+1}) = x_i$ and $v(x_i, x_{i+1}) = x_{i+1}$;
- Type 2, ψ_2: $u(x_i, x_{i+1}) = x_{i+1}$ and $v(x_i, x_{i+1}) = x_i^{-1}$;
- Type 3, ψ_3: $u(x_i, x_{i+1}) = x_{i+1}^{-1}$ and $v(x_i, x_{i+1}) = x_i^{-1}$;
- Type 4, $\psi_{4,h}$: $u(x_i, x_{i+1}) = x_i^h x_{i+1} x_i^{-h}$ and $v(x_i, x_{i+1}) = x_i$;
- Type 5, ψ_5: $u(x_i, x_{i+1}) = x_i x_{i+1}^{-1} x_i$ and $v(x_i, x_{i+1}) = x_i$;
- Type 6, ψ_6: $u(x_i, x_{i+1}) = x_i x_{i+1} x_i$ and $v(x_i, x_{i+1}) = x_i^{-1}$;
- Type 7, ψ_7: $u(x_i, x_{i+1}) = x_i^2 x_{i+1}$ and $v(x_i, x_{i+1}) = x_{i+1}^{-1} x_i^{-1} x_{i+1}$.

Types 1–3 are obviously not faithful, while Types 4–7 are faithful ([14], see also Remark 9.8 in [2] and June 19/2011 addenda in [3] for a useful survey on proofs of faithfulness) and can be used to define link invariants [5, 15]. The Artin representation is a particular case of representation of local type ($\rho_A = \psi_{4,1}$).

Wada conjectured that above families where the only local type representations, up to two symmetries, the involution of the free group F_n sending any generator x_i into its inverse and the involution of the braid group B_n sending any generator σ_j into its inverse: this conjecture was recently proved by Ito [9].

Actually the family of local type representations proposed by Wada is redundant: in [14] was remarked that Type 5 and Type 6 are conjugated and in [5] (Proposition A.1) was proved that type 5 and type 7 were equivalent, more precisely that it exists an automorphism $\chi : F_n \to F_n$ such that $\chi \circ \psi_7(\sigma_i) \circ \chi^{-1} = \psi_5(\mu(\sigma_i))$, $i = 1, 2, \ldots, n-1$, where $\mu : B_n \to B_n$ is the involution sending σ_i into σ_i^{-1}.

Similarly to this result we can prove a relation between Perron-Vannier representations and representations of local type.

Theorem 4.1. *The extended Perron-Vannier representation $\rho_{CP}^{(1)}$ defined in Proposition 3.1 is equivalent to Wada representation ψ_5.*

Proof. Take in the group $F_n = \langle x_1, x_2, \ldots, x_n\rangle$ new basis

$$y_1 = x_1 x_2^{-1},\ y_2 = x_2 x_3^{-1}, \ldots, y_{n-1} = x_{n-1} x_n^{-1},\ y_n = x_n.$$

We have the following action in this basis

$$\psi_5(\sigma_1) : \begin{cases} y_1 \longmapsto y_1, \\ y_2 \longmapsto y_1 y_2, \\ y_j \longmapsto y_j, \qquad j \geq 3, \end{cases}$$

and

$$\psi_5(\sigma_i) : \begin{cases} y_{i-1} \longmapsto y_{i-1} y_i^{-1}, & \\ y_{i+1} \longmapsto y_i y_{i+1}, & \\ y_j \longmapsto y_j, & j \neq i-1, i+1, \end{cases} \qquad i = 2, 3, \ldots, n-1.$$

Define the new representation $\psi_5^- : B_n \longrightarrow \mathrm{Aut}(F_n)$ by the rule

$$\sigma_i \longmapsto (\psi_5(\sigma_i))^{-1}, \ i = 1, 2, \ldots, n-1,$$

then we get

$$\psi_5^-(\sigma_1) : \begin{cases} y_1 \longmapsto y_1, & \\ y_2 \longmapsto y_1^{-1} y_2, & \\ y_j \longmapsto y_j, & j \geq 3, \end{cases}$$

and

$$\psi_5^-(\sigma_i) : \begin{cases} y_{i-1} \longmapsto y_{i-1} y_i, & \\ y_{i+1} \longmapsto y_i^{-1} y_{i+1}, & \\ y_j \longmapsto y_j, & j \neq i-1, i+1, \end{cases} \qquad i = 2, 3, \ldots, n-1.$$

which is exactly Perron-Vannier representation $\rho_{CP}^{(1)}$. □

5. Linear representations, which are induced by the representations of Fenn-Rolfsen-Zhu and Perron-Vannier

Consider the composition of homomorphisms

$$B_n \xrightarrow{\rho_F} \mathrm{Aut}(F_{n+1}) \xrightarrow{\pi} \mathrm{GL}_{n+1}(\mathbb{Z})$$

and denote this composition by $\overline{\rho}_F = \rho_F \circ \pi$, i.e.

$$\overline{\rho}_F : B_n \longrightarrow \mathrm{GL}_{n+1}(\mathbb{Z}).$$

Also, denote

$$r_i = \overline{\rho}_F(\sigma_i), \ i = 1, 2, \ldots, n-1.$$

It is easy to check that

$$r_i = \left(\begin{array}{c|ccc|c} I_{i-1} & & \mathbf{0} & & \mathbf{0} \\ \hline & 1 & 1 & 0 & \\ \mathbf{0} & 0 & -1 & 0 & \mathbf{0} \\ & 0 & 1 & 1 & \\ \hline \mathbf{0} & & \mathbf{0} & & I_{n-i-1} \end{array} \right), \ i = 1, 2, \ldots, n-1,$$

where I_k is the unit matrix of the order k.

Proposition 5.1. *The image of B_n under the homomorphism $\overline{\rho}_F : B_n \longrightarrow \mathrm{GL}_{n+1}(\mathbb{Z})$ is isomorphic to the symmetric group S_n.*

Proof. It is evident that $r_i^2 = I_{n+1}$ for all $i = 1, 2, \ldots, n-1$. Also, since $\overline{\rho}_F$ is a homomorphism then elements r_i are satisfy the braid relations. Hence

$$\overline{\rho}_F(B_n) = \langle \overline{\rho}_F(\sigma_1), \overline{\rho}_F(\sigma_2), \ldots, \overline{\rho}_F(\sigma_{n-1}) \rangle = \langle r_1, r_2, \ldots, r_{n-1} \rangle = S_n.$$

□

Recall that the extended Perron-Vannier representation $\rho^{(2)}_{CP} : B_n \to \mathrm{Aut}(F_{n+1})$, was defined as follows:

$$\rho^{(2)}_{CP}(\sigma_i) : \begin{cases} y_{i-1} \longmapsto y_{i-1} y_i, \\ y_{i+1} \longmapsto y_i^{-1} y_{i+1} \\ y_j \longmapsto y_j, & j \neq i-1, i+1. \end{cases}$$

In this section we will prove that $\rho^{(2)}_{CP}$ is not equivalent to the Artin representation and hence it is not equivalent to the representation of Fenn-Rolfsen-Zhu. To do this define the homomorphism

$$\overline{\rho}_P : B_n \xrightarrow{\rho^{(2)}_{CP}} \mathrm{Aut}(F_{n+1}) \xrightarrow{\pi} \mathrm{GL}_{n+1}(\mathbb{Z})$$

and find the matrix

$$s_i = \overline{\rho}_P(\sigma_i), \ i = 1, 2, \ldots, n-1.$$

We see that

$$s_i = \left(\begin{array}{c|ccc|c} I_{i-1} & & \mathbf{0} & & \mathbf{0} \\ \hline & 1 & 0 & 0 & \\ \mathbf{0} & 1 & 1 & -1 & \mathbf{0} \\ & 0 & 0 & 1 & \\ \hline \mathbf{0} & & \mathbf{0} & & I_{n-i-1} \end{array} \right).$$

Hence

$$\overline{\rho}_P(B_n) = \langle s_1, s_2, \ldots, s_{n-1} \rangle.$$

The following lemma is trivial:

Lemma 5.2. *For any integer k holds*

$$s_i^k = \left(\begin{array}{c|ccc|c} I_{i-1} & & \mathbf{0} & & \mathbf{0} \\ \hline & 1 & 0 & 0 & \\ \mathbf{0} & k & 1 & -k & \mathbf{0} \\ & 0 & 0 & 1 & \\ \hline \mathbf{0} & & \mathbf{0} & & I_{n-i-1} \end{array} \right), \ i = 1, 2, \ldots, n-1.$$

In particular, s_i has infinite order.

Using this lemma we can prove

Proposition 5.3. *The representations $\rho^{(2)}_{CP}$ and $\widetilde{\rho}_A$ are not equivalent.*

Proof. Assume, that they are equivalent. Hence, there is an automorphism $\psi \in \mathrm{Aut}(F_{n+1})$ such that

$$\psi^{-1} \widetilde{\rho}_A \psi = \rho^{(2)}_{CP}.$$

In particular,

$$\psi^{-1} \widetilde{\rho}_A(\sigma_i) \psi = \rho^{(2)}_{CP}(\sigma_i), \ i = 1, 2, \ldots, n-1.$$

This is an equality in $\mathrm{Aut}(F_{n+1})$ and hence under the action of the homomorphism

$$\pi : \mathrm{Aut}(F_{n+1}) \longrightarrow \mathrm{GL}_{n+1}(\mathbb{Z})$$

it goes to an equality

$$\pi(\psi^{-1}) \pi(\widetilde{\rho}_A(\sigma_i)) \pi(\psi) = \pi(\rho^{(2)}_{CP}(\sigma_i))$$

in $\mathrm{GL}_{n+1}(\mathbb{Z})$. Since $\pi(\widetilde{\rho}_A(\sigma_i)) = r_i$, $\pi(\rho^{(2)}_{CP}(\sigma_i)) = s_i$ then we have the equality

$$\pi(\psi^{-1}) \, r_i \, \pi(\psi) = s_i$$

for some matrix $\pi(\psi) \in \mathrm{GL}_{n+1}(\mathbb{Z})$. But we know that $r_i^2 = I_n$ and s_i has infinite order. Hence this equality doesn't hold. □

We end this section with a Burau-like representation associated to ρ_F: more precisely we will associate to

$$\rho_F : B_n \longrightarrow \mathrm{Aut}(F_{n+1})$$

a linear representation

$$\overline{\rho}_F : B_n \longrightarrow \mathrm{GL}_{n+1}(\mathbb{Z}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]).$$

We will use the Magnus representation [4].

Define a ring homomorphism

$$\tau : \mathbb{Z}F_{n+1} \longrightarrow \mathbb{Z}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}],$$

by the rule $\tau(y_i) = t_i$, $i = 0, 1, \ldots, n$, and extending by linearity.

To use the Magnus representation the following equations must be true

$$\tau(y_i^{\rho_F(\sigma_j)}) = \tau(y_i),\ i = 0, 1, \ldots, n,\ j = 1, 2, \ldots, n-1.$$

If $i \neq j$ then

$$y_i^{\rho_F(\sigma_j)} = y_i$$

and our equation is true. If $i = j$ then we have

$$\tau(y_j^{\rho_F(\sigma_j)}) = \tau(y_{j-1} y_j^{-1} y_{j+1}) = t_{j-1} t_j^{-1} t_{j+1}.$$

On the other hand $\tau(y_j) = t_j$. Hence we have the system of equations

$$t_{j-1} t_j^{-1} t_{j+1} = t_j,\ j = 1, 2, \ldots, n-1.$$

From this system we find

$$t_j = \sqrt[n]{t_0^{n-j} t_n^j},\ j = 1, 2, \ldots, n-1.$$

The linear representation $\overline{\rho}_F$ maps any element from B_n to an automorphism of free $n+1$-dimension $\mathbb{Z}[t_0^{\pm}, t_1^{\pm}, \ldots, t_n^{\pm}]$-module with basis $\{v_0, v_1, \ldots, v_n\}$. A braid $\beta \in B_n$ maps to automorphism

$$\overline{\rho}_F(\beta) : v_i \longmapsto \sum_{j=0}^{n} \tau\left(\frac{\partial y_i^{\rho_F(\beta)}}{\partial y_j}\right) v_j,\ i = 0, 1, \ldots, n.$$

It is evident that is enough define the automorphisms $\overline{\rho}_F(\sigma_k)$, $k = 1, 2, \ldots, n-1$. We will write $y_i^{\sigma_k}$ instead of $y_i^{\rho_F(\sigma_k)}$. Calculating the Fox derivatives.

We see that if $i \neq k$ then $y_i^{\sigma_k} = y_i$ and

$$\frac{\partial y_i^{\sigma_k}}{\partial y_j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

If $k = i$ then we have

$$y_i^{\sigma_i} = y_{i-1} y_i^{-1} y_{i+1}$$

and

$$\frac{\partial y_i^{\sigma_i}}{\partial y_j} = 0,\ j \neq i-1, i, i+1;$$

$$\frac{\partial y_i^{\sigma_i}}{\partial y_{i-1}} = 1,\ \frac{\partial y_i^{\sigma_i}}{\partial y_i} = -y_{i-1} y_i^{-1},\ \frac{\partial y_i^{\sigma_i}}{\partial y_{i+1}} = y_{i-1} y_i^{-1}.$$

Applying the map τ, we will have

$$\overline{\rho}_F(\sigma_i) : \begin{cases} v_i \longmapsto v_{i-1} - t_{i-1}t_i^{-1}v_i + t_{i-1}t_i^{-1}v_{i+1}, & \\ v_k \longmapsto v_k, & k \neq i, \end{cases}$$

for all $i = 1, 2, \dots, n-1$.

Since $t_j = \sqrt[n]{t_0^{n-j}t_n^j}$ then

$$t_{i-1}t_i^{-1} = \sqrt[n]{t_0^{n-(i-1)}t_n^{i-1}} / \sqrt[n]{t_0^{n-i}t_n^i} = \sqrt[n]{t_0/t_n}.$$

Hence, if we define $t = \sqrt[n]{t_0/t_n}$ then we have:

Proposition 5.4. *There exists a linear representation*

$$\overline{\rho}_F : B_n \longrightarrow \mathrm{GL}_{n+1}(\mathbb{Z}[t^{\pm 1}]),$$

which is defined as follows:

$$\overline{\rho}_F(\sigma_i) : \begin{cases} v_i \longmapsto v_{i-1} - tv_i + tv_{i+1}, & \\ v_k \longmapsto v_k, & k \neq i, \end{cases}$$

for all $i = 1, 2, \dots, n-1$.

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