LEOPOLDT’S CONJECTURE FOR CM GALOIS EXTENSIONS

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ABSTRACT. We start by using Baker theory for proving the simple result, that Leopoldt’s conjecture is valid for galois extensions $K/\mathbb{Q}$ with solvable groups. In particular, if Leopoldt’s conjecture is false for $K$, then $p$ is split in $K$.

Let now $K$ be a galois totally real extension and $M/K$ be the product of $\mathbb{Z}_p$ extensions of $K$ which intersect the cyclotomic $\mathbb{Z}_p$ extension $K_\infty/K$ in $K$ and let $P = \{\mathcal{O}(K) \supset \wp \supset (p)\}$ be the set of primes above $p$. Then $M = K$ is equivalent to Leopoldt’s conjecture, and if $M \neq \mathbb{Q}$, we first show that the galois groups of localizations $G_\wp = \text{Gal}(M_\wp/K_\wp)$ – which are $\mathbb{Z}_p[D_\wp]$ - modules, with $D_\wp \subset \text{Gal}(K/\mathbb{Q})$ the decomposition of $\wp$ – have only the trivial annihilator in this group ring.

As a consequence, it follows that there is a norm coherent sequence of ramified ideals $\wp_n \subset K_n$ in the cyclotomic $\mathbb{Z}_p$ - extension, with diverging orders in the respective class groups. This contradicts a special case of the conjecture of Greenberg, namely the following: let $K$ be a totally real galois number field, $K_\infty$ the $\mathbb{Z}_p$ - cyclotomic extension and $B^+ \subset A$ the $\Lambda$ - submodule of the projective limit of $p$ - parts of the ideal class groups $A_n \subset C(K_n)$ which are generated by ramified primes in $A_n$. Then $B^+$ is finite.

I sketch briefly the proof: if $K'_n = K_n[\zeta]$ contain the $p^n$ -th roots of unity, so $K'_n$ are CM, and $M_n \supset K'_n$ is the maximal product of $\mathbb{Z}_p$ extensions of $K'_n$, then I first show that

$$K'_\infty \cdot M'_n \subset \Omega_E,$$

where $\Omega_E$ is the injective limit of Kummer extensions generated by roots of the units in $K'_m$, $m \geq n$. If $b = (b_n)_{n \in \mathbb{N}} \in (B')^+$ is a sequence containing ramified ideals $\wp_n \subset K'_n$ and $\wp_n^{p^{n+k}} = (\pi_0)$ is a principal ideal, then $K'_n[\zeta_0^{1/p^n}] \subset M'_n$ and the special case of Greenberg presented above implies that the orders of $\wp_n$ are bounded when $n \to \infty$.

Date: Version 1.0 February 23, 2009.
Contents

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