Variations on the Fibonacci binary sequence

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The Fibonacci binary sequence is defined as follows. First consider the substitution rule where 0 is replaced by 01, and 1 is replaced by 0. Applying this substitution rule to a finite string of symbols 0 and 1 means applying it "in parallel" to each 0 and each 1 of the string. For example, applying it to 001 gives 01010 (the first two zeroes where replaced by 01 while simultaneously the last 1 was replaced by 0). Then start from 0 and apply iteratively the substitution rule; you obtain successively

0
01
010
01001
01001010
...

Note that the length of the strings you obtain is $1, 2, 3, 5, 8, \ldots$ where you might recognize one of your favorite sequences, and that the sequence of strings "converges" in a reasonable sense to an infinite sequence

$$\mathbf{F} = 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ \ldots$$

which is thus called the Fibonacci binary sequence.

We will see how this sequence is in some sense one of the "simplest" non-periodic sequences, how it can be obtained by playing billiard on a square and what other sequences share these two properties. Finally we will indicate unexpected inequalities involving these sequences and their shifts; for example if we let S denote the shift on sequences, so that $S\mathbf{F} = 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ \ldots$, $S^2\mathbf{F} = 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ \ldots$, and if $\leq$ is the lexicographical order on binary sequences, then for all $k \geq 1$

$$0\ \mathbf{F} \leq S^k(\mathbf{F}) \leq 1\ \mathbf{F}.$$